ELEMENTS OF DYNAMIC

Phys.
Mech
C.

ELEMENTS OF DYNAMIC

AN INTRODUCTION TO THE STUDY OF

MOTION AND REST

IN SOLID AND FLUID BODIES

BY

W. K. CLIFFORD, F.R.S.

LATE PROFESSOR OF APPLIED MATHEMATICS AND MECHANICS AT
UNIVERSITY COLLEGE, LONDON.

PART I. KINEMATIC. BOOK IV. AND APPENDIX.

London:
MACMILLAN AND CO.
AND NEW YORK.
1887

Cambridge:
PRINTED BY C. J. CLAY, M.A. AND SONS,
AT THE UNIVERSITY PRESS.

PREFACE.

I HAVE sufficiently explained in my letter to Mrs Clifford (see "Mathematical Papers") the reasons which led me to accept the responsibility of editing the following fragments. A few words as to the fragments themselves may not be out of place here.

The first 56 pages are contained in 43 pages of MS. These are carefully written out and paged, and in the form in which they are left may be considered as nearly representing that in which they would have been given to the world by Clifford himself.

Pages 57 to 72 consist of detached portions of manuscript written out in Clifford's usual careful manner, and were evidently intended, after a further examination, to take their places in his book. The remainder of Appendix I. is printed here mainly with the view of showing Clifford's work in its early stage. Thus (C) on the "Top" is in its present form almost, if not quite, unintelligible: most probably Clifford intended to discuss the subject in connection with the "Kinetic analogy" of Kirchhoff.

In Appendix II. I reprint the "Syllabus of Lectures on Motion" from the "Papers" (pp. 516—524), chiefly because it contains the article on Fourier's theorem which was promised in the "Dynamic," p. 37: and the

"Abstract of the Dynamic" because it passes with clear and rapid touch over the subject as expounded in the already published volume. The two "contents" (C) and (D) put the reader in possession of what it was the Author's intention to discuss had he lived to complete his work.

I have not hesitated to extract from the Examination-papers set by Clifford at University College a number of questions, very characteristic of the Author, and to arrange them as well as I could under the respective chapters: in this course I have already met with warm approval.

I may mention that there should be added to my "Bibliographical account" in the "Papers," a reference to notes of a lecture on "Energy and Force," delivered by Clifford before the Royal Institution on March 28, 1873. Notes of this lecture, taken by Mr F. Pollock, and revised by Mr J. F. Moulton, F.R.S., are published in *Nature*, Vol. XXII. p. 123 (June 10, 1880). After consulting with two or three mathematicians upon whose judgment I could thoroughly rely, I have decided not to insert these notes in the present volume. Clifford had commenced an Index and had proceeded sufficiently far to allow one to see on what lines he would have completed it: this task I have fulfilled on his lines.

R. TUCKER.

CONTENTS.

BOOK IV. MASSES.

CHAPTER I. THE MASS-CENTRE.

CHAPTER II. SECOND MOMENTS.

CHAPTER III. MOMENTUM.

APPENDIX I. (A)

(B)

(C)

(D)

(E)

(F)

APPENDIX II. (A)

(B)

(C)

(D)

APPENDIX III.

The references to the "Dynamic" Part I. are enclosed throughout in square brackets [ ].

The following errata occur in that Volume:

p. 24, l. 13, read $\epsilon_1 - \epsilon_2$;
p. 102, 11 up, for $-$ read $=$;
p. 103, l. 6, read an^2;
p. 131, 9 up, for $\frac{1}{4}$ read $\frac{1}{2}$;
p. 132, l. 3, for $\cos\theta$ read $\cot\theta$;
,, l. 6, insert $-$ before h bis;
,, l. 8, for λ read h.

BOOK IV. MASSES.

CHAPTER I. THE MASS-CENTRE.

DENSITY.

WE have seen how to measure a change in the size or volume of a body. When the size of a body is diminished, it becomes more closely packed together, or more *dense;* when the size is increased, it becomes less dense. Suppose that in a certain arbitrary state of the body we reckon its density to be unity, then when it is compressed into one-nth of the volume its density will be n times as great. Or, if v is the volume of that which, at density 1, filled a unit of volume, its density is now $\frac{1}{v}$. The density of a body may be different in different parts; the density of the air, for example, diminishes as we go upwards. The question then arises, how are we to compare different portions of the same substance, so as to find out whether they are of the same or different densities? Given two samples of air in bottles, or two samples of iron, one of which has been hammered, how shall we compare their densities?

The answer is, that we must take equal volumes of the two samples, and measure the *quantity of stuff* that there is in each. For the two samples of air, we may put them into perfectly flexible air-tight bags, so as not to fill the bags; then when these bags are held freely in the atmosphere at the same level, the quantities of air are proportional to the volumes they occupy. The two samples of

iron may be melted, and their volumes compared in that state. For other substances the comparison by such methods might be more difficult.

If a piece of stuff is of uniform density, the quantity of stuff in it is the product of the volume and the density, provided that the unit of quantity is taken to be that of a unit of volume at unit density. The quantity of stuff in a piece is called the *mass* or measure of that piece.

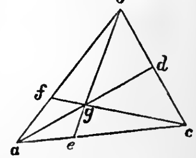
We shall give to the word *mass* a more extended meaning when we come to consider the laws of motion*; and shall then explain much easier methods of comparing the masses of two pieces of the same stuff, as well as (in the extended sense) of two pieces of different stuffs. For the present, however, we shall suppose all the bodies spoken of to be made of the same stuff, and we shall mean by the mass of a given portion merely the quantity of that stuff which it contains. All the results we shall get will be applicable to the more extended meaning of the word.

When the density varies from point to point, the density *at* any point is the mass which a unit of volume would have if its density were uniformly equal to that at the point†.

MASS-CENTRE.

If a particle of mass m be situated at a point p, the vector $m \,.\, op$ is called the *mass-vector* of the particle from the origin o.

If a mass l be at a and a mass m at b, a mass $l+m$ at a point f such that $$l \,.\, fa + m \,.\, fb = 0$$ shall be called the *resultant* of the two masses.



Since we know that [p. 8]
$$l \,.\, oa + m \,.\, ob = (l+m)\, of,$$
it follows that the mass-vector of the resultant mass is

* [See below, p. 58.]

† [This sentence is hardly satisfactory, especially without any reference to the doctrine of limits.]

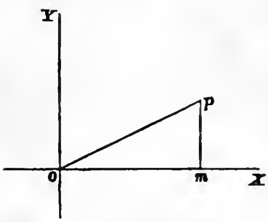
equal to the sum of the mass-vectors of the components, from any origin.

If there be a mass n at c, the resultant of $l+m$ at f and n at c will be called the resultant of l at a, m at b, n at c, and so on for any number of particles. It follows from the general theorem already proved that in all cases the mass-vector of the resultant mass is the sum of the mass-vectors of the components, from any origin.

The position of the resultant mass is called the *centre of mass* or *mass-centre* of the given particles.

The *moment* of a particle in regard to any line or plane is the product of the mass of the particle by its distance from the line or plane.

The moment of the resultant mass is equal to the sum of the moments of the components on any line or plane. For let the origin o be taken in the given line or plane, oX or oXZ; then the moment of the particle l at p is equal to the component of its mass-vector $l \, . \, op$ perpendicular to the line or plane, namely, $l \, . \, mp$. And since the mass-vector of the resultant mass is equal to the sum of the mass-vectors of the component masses, it follows that its component perpendicular to any line or plane is the sum of their components [p. 12].

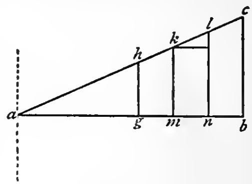


MASS-CENTRE OF ROD.

If a mass be distributed uniformly along a straight line ab, the mass-centre is at the middle point g of the line; for we may divide the line into pairs of particles equidistant from g, so that each pair has g for its mass-centre.

We shall now verify that the moment of the resultant mass is equal to the moment of the rod in regard to any line through a perpendicular to ab.

Let bc, perpendicular to it, be equal in length to ab multiplied by the mass of a unit of length of it. Join ac, suppose the length ab divided into small portions of which mn is one, and draw mk, nl perpendicular to ab meeting ac in k, l. Then the moment of mn in regard to a line through a perpendicular to the rod will lie between am multiplied by the mass of mn, and an multiplied by the same mass. Now the moment of the mass of a unit of length at m is mk, and at n is nl. Hence the moment of mn lies between $mn \,.\, mk$ and $mn \,.\, nl$. Thus the moment of ab lies between two values which include the area abc and which can be made as nearly equal as we like by increasing the number of parts into which ab is divided. That is to say, the moment of ab is equal to the area abc, namely to



$$\tfrac{1}{2}\, ab \,.\, bc = ab \,.\, gh,$$

where g is the middle point of ab. Now gh is the moment of the mass of a unit of length at g; therefore $ab \,.\, gh$ = moment of the mass of ab collected at g.

In the same way it appears that the moment of a portion of the rod, such as mb, is equal to the area $mbck$ which stands over it.

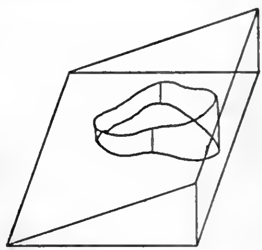
In the case of a thin plate or lamina in the form of a parallelogram, such that the masses of any two equal areas of it are equal, the centre of mass is at the intersection of the diagonals, which is also the intersection of the lines joining the middle points of opposite sides (*median* lines). For the area may be divided into thin strips by lines parallel to one side, each of which has its mass-centre on a median line.

And since a parallelepiped may be divided into such thin plates by planes parallel to two of its faces, the resultant masses of these will all lie in the straight line ab joining the centres of those faces, if the density of the solid be uniform (i.e. the masses of any two equal volumes equal); and they will be distributed uniformly along this

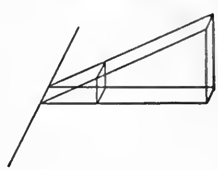
line, since equal lengths of it will represent equal slices of the solid; therefore the centre of mass of the parallelepiped is at the middle point g of ab.

TRIANGLE AND TETRAHEDRON.

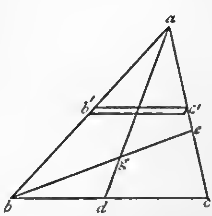
In general, the moment of any plane lamina of uniform density (masses of equal areas equal) about any line in its plane is the volume of a solid standing on the lamina, bounded by lines through its boundary perpendicular to its plane, and by a plane drawn through the straight line, such that the height of every point of it is equal to the moment about the line of the mass of a unit of area situate directly under the point.



The proof of this is precisely similar to that of the case of a uniform rod. The lamina is to be divided into thin strips by lines perpendicular to the given line, and it is proved as in that case that the moment of each of these strips is the part of the volume belonging to it.



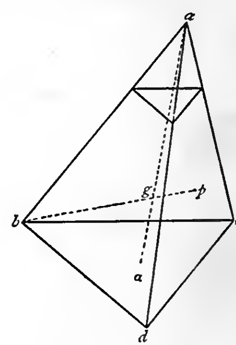
If a plane lamina is such that all chords of it parallel to a given direction are bisected by a certain straight line, then the centre of mass is in that straight line. For the lamina may be divided into thin strips like $b'c'$, which by cutting off small pieces at the end, may be made into parallelograms whose centre of mass is in ad. The whole mass of the pieces cut off may be made as small as we like by increasing the number and diminishing the breadth of the strips. Consequently the mass-centre must be in ad.



In the case of the triangle, for example, the mass-centre is in *ad*, and also in *be*; therefore it is in their intersection *g*, which is also the mass-centre of three equal particles at *a*, *b*, *c*, and one-third of the way from *d* towards *a*.

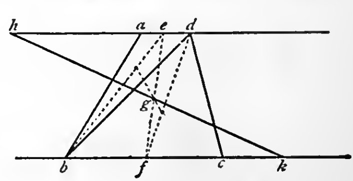
So again, if all the sections of a solid by a series of parallel planes have their mass-centres in the same straight line, the mass-centre of the solid, supposed to be of uniform density, is in that line. For we may divide the solid into thin slices by such parallel planes, which by cutting off small pieces at their boundaries may be made of uniform surface-density, and therefore have their resultant masses on the given line. The mass of the pieces cut off may be made as small as we like by increasing the number and diminishing the thickness of the slices.

Thus, in the case of a tetrahedron, all plane sections parallel to the face *bcd* have their mass-centres on the line *aα*, joining the vertex *a* to the mass-centre *α* of *bcd*. Hence the mass-centre of the tetrahedron is in *aα*. Similarly it is in *bβ*, and therefore in their intersection *g*. Hence it coincides with the mass-centre of four equal particles at *a*, *b*, *c*, *d*, and is therefore one-quarter of the way from *α* towards *a*. It is also the middle point of the three lines which join the middle points of opposite edges. [See p. 10.]



QUADRILATERALS.

To find the mass-centre of a *trapezium*, or quadrilateral with two sides *ad*, *bc*, parallel; we observe, first, that it must be in the line *ef* joining the middle points of these sides, since this line bisects all chords parallel to them. Next,

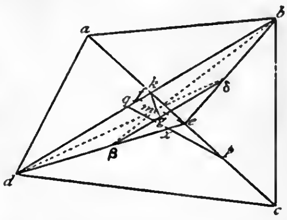


the trapezium being composed of the triangles *adb*, *bdc*, its mass-centre must be in the line joining their mass-centres, which are one-third way from *e* towards *b* and from *f* towards *d* respectively; and it must divide this line in the inverse ratio of those triangles, that is, as $bf : ae$, or say as $b : a$. Hence the *middle third* of *ef* must be divided in this ratio in *g*. The two parts being represented by *b* and *a*, the third of *ef* is represented by $a+b$; therefore *ge* is represented by $(a+b)+b$ or $a+2b$, and *fg* by $a+(a+b)$ or $2a+b$. Hence

$$eg : gf = a+2b : 2a+b.$$

Make then *ah* equal to *cb*, and *ck* equal to *ad*, then *hk* will meet *ef* in the mass-centre *g*.

In a quadrilateral *abcd* of any shape, let *k*, the intersection of the diagonals *ac*, *bd*, be called the *cross-centre*, and *m*, the middle of the line joining their middle points *e*, *f*, the *mid-centre* (mass-centre of four equal particles at *a*, *b*, *c*, *d*). The mass-centre β of the triangle *acd* or of three equal particles at *a*, *c*, *d*, is in *bm*, so that $m\beta = \frac{1}{3}bm$. Similarly δ the mass-centre of *abc* is in *dm* so that $m\delta = \frac{1}{3}dm$. We have to divide $\beta\delta$ in the inverse ratio of these triangles, that is, as $dk : kb$. Hence the point required is where *km* meets $\beta\delta$, and consequently

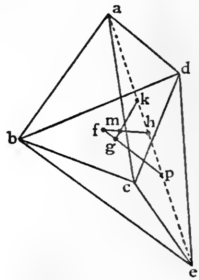


$$mg = \tfrac{1}{3}km.$$

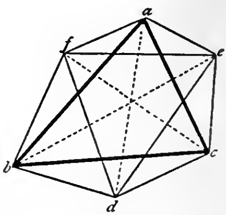
If we take $ep = ke$, and $fq = kf$, so that *p*, *q* are the reflexions of *k* on the diagonals, *g* is mass-centre of the triangle *kpq*. For it is on a line through *e* dividing $\beta\delta$ and therefore *db* in the ratio $bk : kd$, that is, on *eq*. Similarly it is on *pf*.

PLANE-FACED SOLIDS.

The *triangle-faced pentacron* is a solid made of two tetrahedra with a common base. The intersection k of the diagonal ae and the diagonal plane bcd is the *cross-centre*. Let p be the reflexion of k on ae, so that $pe = ak$, and f the mass-centre of bcd; then g the mass-centre of the solid is one-quarter way from f to p. For the mass-centres α and ϵ of the tetrahedra $ebcd$, $abcd$ are one-quarter way from f to e and a respectively, and fg divides $\alpha\epsilon$, and therefore ea, in the ratio $ek : ka$. Consequently it passes through p. If m be the mid-centre (mass-centre of equal particles at a, b, c, d, e; it is $\frac{2}{5}$ way from f to h the middle of ae) we know that $m\alpha = \frac{1}{4}am$, $m\epsilon = \frac{1}{4}em$; hence $mg = \frac{1}{4}km$, or the mass-centre of the solid is in the prolongation of the line joining the cross- and mid-centres, at a distance from the latter equal to one-quarter of the distance between them.



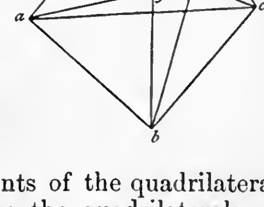
The *octahedron abcdef* is one form of triangle-faced hexacron and may be regarded as made of two pyramids *abcef*, *dbcef* standing on a common skew quadrilateral base *bcef*. There are three such quadrilaterals, the other two being *aedb*, *acdf*. It is to be understood that, in general, no two of the diagonals *ad*, *be*, *cf* intersect, so that no one of these quadrilaterals is plane. But the *middle points* of the sides of a skew quadrilateral are always in one plane; for (e.g.) the line joining the middles of *bf*, *fe*, and the line joining the middles of *bc*, *ce*, are both parallel to *be*, and any two parallel lines are in one plane. To each of the three quadrilaterals there is such a plane, and the intersection k of these planes is called the *cross-centre*. The



mid-centre m, or mass-centre of equal particles placed at the vertices, is the mass-centre of the middle points of the three diagonals. Now the solid is the sum of the four tetrahedra $adef$, $adfb$, $adbc$, $adce$, and therefore its mass-centre g is in the plane containing their mass-centres. Now the mass-centre of $adef$, say x, being also the mass-centre of four equal particles at a, d, e, f, is on the line joining the mid-centre with the middle point p of bc, so that $xm = \frac{1}{2}mp$. Hence the plane through the mass-centres x, y, z, w of the four tetrahedra just mentioned, is parallel to the plane through the middle points $pqrs$ of bc, ce, ef, fb, and at half the distance from m of that plane. Each of the diagonals gives rise to such a division of the solid into tetrahedra, and it follows that the mass-centre g lies on each of three planes parallel to the three planes which intersect in the cross-centre k and at half the distance from m. Hence g is on the line km, so that

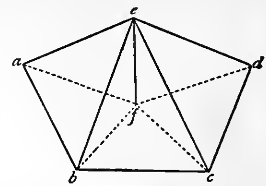
$$mg = \frac{1}{2}km.$$

A particular case of this solid is the *frustum of a tetrahedron*, one vertex of which is cut off by a plane section. It occurs when the faces afe, abf are in one plane, as also bdf, bcd, and ced, cae. We may either count (as is here done) af, bd, ce for edges of the solid, and ad, be, cf for diagonals; or else we may take ad, be, cf for edges, and af, bd, ce for diagonals. In the former case the cross-centre is the intersection of planes through the middle points of the quadrilaterals $becf$, $cfad$, $adbe$; in the latter case the quadrilaterals are $bdce$, $ceaf$, $afbd$. The cross-centre is of course the same point in either case, so that these six planes intersect in a point k. The position of g is given as in the general case by



$$mg = \frac{1}{2}km.$$

The other form of triangle-faced hexacron is shewn in the figure. Each of the vertices e and f has *five* edges through it, b and c have four, a and d three; while in the octahedron every vertex has four edges through it. The construction of the cross-centre is not quite so simple in this case. Let p be a point one-fourth of the way from the middle of ac to the middle of cd, and q a point one-fourth of the way from the middle of bd to the middle of ab. Through p draw a plane parallel to cef, and through q a plane parallel to bef. The intersection of these planes with a plane through the middle points of ba, ad, dc is the cross-centre k. If m be the mid-centre and g the mass-centre, then as before we have $mg = \frac{1}{2}\, km$.



To prove this we observe that the solid is the sum of the four tetrahedra $efab$, $bdec$, $bdcf$, $bdfe$, and that their mass-centres are on straight lines through the mid-centre and the middle points of cd, af, ae, ac respectively, at half the distance of these latter from m. The middles of af, ae, ac are in a plane parallel to efc at half its distance from a. The mass-centre of these points and the middle of cd is one-fourth the way from a point in this plane to the middle of cd. It is therefore in a plane parallel to efc through p, which is one-fourth the way from the middle of ac to the middle of cd. Similarly by dividing the solid into the tetrahedra $efab$, $efbc$, $efcd$, we may shew that k is in a plane through the middles of ba, ad, dc.

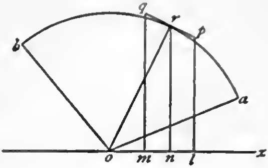
The method of the preceding five paragraphs, the useful names of the mid-centre and cross-centre, and the theorem for the tetrahedral frustum are due to Sylvester*.

CIRCULAR ARC AND SECTOR.

The moment of a *circular arc* in regard to any line through the centre is equal to its projection multiplied by

* *Phil. Mag.* [Vol. XXVI., pp. 167—183 (1863). See *Math. Papers*, p. 409, or *Proc. of Lond. Math. Soc.* Vol. IX. p. 28.]

the radius. Let pq be a tangent at r, or perpendicular on it from the centre, m, n, l the projections of q, r, p. Then the moment of pq is



$$pq \, . \, rn = ml \, . \, or,$$

since

$$ml \; : \; pq = rn \; : \; or = \sin rox.$$

Thus the moment of every piece of straight line is equal to its projection multiplied by its perpendicular distance from the origin. If we draw a polygon circumscribing the circular arc, the distance of all its sides from the origin is equal to the radius of the circle; and therefore its moment is the radius multiplied by its projection. Such a polygon may be made to approximate as nearly as we like to the circle by increasing the number of sides and diminishing their length; therefore the same thing is true for the circle.

Taking now ox parallel to the chord of the arc, we find the distance of its mass-centre from o to be

$$\text{radius} \times \text{chord } ab \; : \; \text{length of arc},$$

or if the angle $aob = 2\theta$, radius $= a$, then this distance is

$$a \sin \theta \; : \; \theta.$$

A circular sector may be approximately divided into small triangles whose vertex is at o and whose mass-centres are distant $\frac{2}{3}a$ from o. Hence it is equivalent to a uniform arc of two-thirds the radius, and the distance from o of the mass-centre is $2a \sin \theta \; : \; 3\theta$. Thus in the case of a semi-circle, $\theta = \frac{1}{2}\pi$, distance of mass-centre from o

$$= 4a \; : \; 3\pi.$$

ROD OF VARYING DENSITY. APPLICATIONS.

A rod whose density varies as the distance from one end is equivalent to a uniform triangle with its base bisected by the other end, and therefore its mass-centre is $\frac{1}{3}$ of the length from the other end. If the density varies as the square of the distance from one end, the rod may be regarded as a uniform tetrahedron whose base has its mass-

centre at the other end; consequently the mass-centre is one-fourth of the length from the other end. Generally, suppose the density to vary as the kth power of the distance x from one end, a. Then the mass of a small length δx, one point of which

is at distance x from a, would be $x^k \delta x$ if the density in δx were uniformly what it is at the distance x. Thus $\Sigma x^k \delta x$ is an approximation to the mass of the rod, which can be made as close as we like by diminishing the δx and increasing their number. Hence the mass of the rod

$$= \int_0^a x^k dx = a^{k+1} \ : \ k+1.$$

Similarly the moment of δx about a is approximately

$$x \,.\, x^k \delta x \text{ or } x^{k+1} \delta x,$$

and therefore the moment of the rod is

$$\int_0^a x^{k+1} dx = a^{k+2} \ : \ k+2.$$

Therefore distance of mass-centre

$$= a\,(k+1) \ : \ k+2,$$

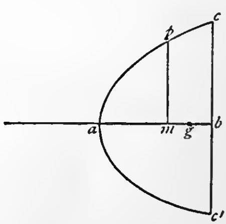
or, from the other end,

$$= a \ : \ k+2.$$

Thus the two examples given are cases of this general rule:—In a rod whose density varies as the kth power of the distance from one end, the mass-centre is one $(k+2)$th part of the length from the other end.

For example, the *parabolic segment* acc' is equivalent, since pm varies as $\sqrt{am}$, to a rod ab whose density varies as the square root of the distance from the end a; so that in this case $k=\frac{1}{2}$. Hence

$$ag = \tfrac{3}{5} ab.$$



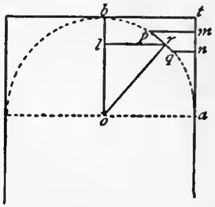
This result is useful, because any small segment of a curve may be treated approximately as a parabolic segment. We observe also

that the area is $\frac{2}{3}\,ab\,.\,cc'$, a result first obtained by Archimedes in a different manner.

In a *paraboloid*, which is the solid got by spinning a parabola about its axis, and giving a homogeneous strain to the result, the area of a section parallel to the tangent plane at a is proportional to mp^2, and therefore to am; thus the solid is equivalent to a uniform triangle acc', its volume is one-half that of the including cylinder, and the mass-centre is two-thirds way from a to b.

SURFACE AND VOLUME OF HEMISPHERE.

We go on to consider the area and mass-centre of a *portion of a spherical surface cut out by two parallel planes.* While the circle ab, by revolving about ob, generates a sphere, the tangent at by the same revolution will generate a right circular cylinder. Let pq be a small tangent to the circle, mn its projection on at, l the projection on ob of the point of contact r. Then the area traced out by pq will be equal to pq multiplied by the length of the path of r (r being its middle point); that is, it is $2\pi\,.\,pq\,.\,lr$. Similarly the area traced out by mn will be $2\pi\,.\,mn\,.\,oa$. Now



$$mn : pq = lr : or = lr : oa,$$

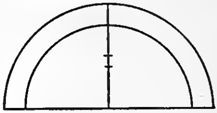
so that these two areas are equal.

Since then an arc of the circle may be indefinitely approximated to by a circumscribed polygon, it follows that the area between any two parallel plane sections of the spherical surface is equal to the area between the same planes on the circumscribing cylinder whose axis is perpendicular to them. And since the small strips of which these are the sums are respectively not only *equal* but *equivalent* (having the same mass-centres on ob) it follows also that the mass-centre of the spherical area is midway

between the two planes of section. Thus the mass-centre of a hemispherical surface is distant half a radius from the centre.

We learn also, in passing, that the whole area of the sphere is equal to that of the curved surface of the cylinder, and is therefore $=4\pi a^2$. (Archimedes.)

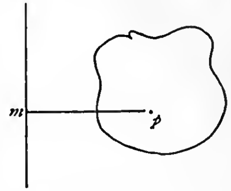
To determine the mass-centre of a *hemisphere*, we may regard it as made up of thin concentric spherical shells. The mass-centre of every shell is distant half its radius from the centre, and the mass of the shell, if all are of the same thickness, varies as the square of the radius. Hence the hemisphere is equivalent to a straight rod half a radius long, whose density varies as the squared distance from the centre; and whose mass-centre is accordingly distant from the centre $\frac{3}{8}$ of a radius.



CHAPTER II. SECOND MOMENTS.

PLANE AREA.

If the density of an area is proportional to the distance from a line in its plane, being reckoned positive on one side of the line and negative on the other, the line is called a *neutral axis;* the mass-centre of the area, having that density, is called the *pole* of the line in regard to the area; and the moment of it in regard to the line is called the *second moment* of the *uniform* area in regard to the line, or of the line in regard to the uniform area. The area may be said to be *loaded from* the line.



If we consider a small area δa in the neighbourhood of the point p, and draw pm perpendicular to the given line, the *moment* (or, as we may now call it for distinction, the *first moment*) of δa in regard to the line is approximately $mp \cdot \delta a$, or $x\delta a$. If however the density is proportional to the distance from the line, or say the density is kx, then the mass of δa is approximately $kx\delta a$, and its moment in regard to the line is $kx^2\delta a$. Thus the *second moment* of the area is what $\Sigma kx^2\delta a$ indefinitely approximates to with diminution of the δa; that is, it is $\int kx^2 da$ taken over the whole area. The second moment of any number of particles in regard to a line is the sum of their masses, each multiplied by the square of its distance from the line.

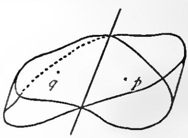
The distance of the *pole* of the line from it is the moment of the loaded area divided by its mass, or

$$\int kx^2 d\alpha : \int kx d\alpha, = \int x^2 d\alpha : \int x d\alpha,$$

which is the ratio of the second moment to the first.

The *loaded area* may be represented, in the same way as the first moment in regard to a line, by a solid standing upon the plane, bounded by straight lines normal to the plane through the boundary of the area, and by an oblique plane through the neutral axis. Parts of this solid which are above the plane are positive, parts below it are negative. The second moment of the area in regard to the line is the first moment of this solid in regard to a normal plane through the line.

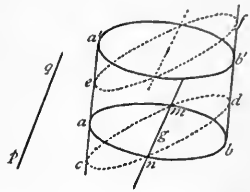
If the line pass through the mass-centre of the unloaded or uniform area, the loaded area is of zero mass, there being as much of it on one side of the line as the other. Let the mass-centre of one side be p, and of the other side q; then we have a mass m at p and a mass $-m$ at q. It is easy to see that the moment of the system about all lines in the plane having the same direction is constant, being $m \cdot pq \cdot \sin\theta$, where θ is the angle they make with pq.



PARALLEL AXES. SWING-RADIUS.

The second moment of an area in regard to any line is equal to the second moment about a parallel line through the mass-centre, together with the second moment in regard to the first line of the whole area if collected at the mass-centre. Let pq be the first line, mn parallel through the mass-centre g. The second moment about mn is the first moment of the two solids $mnac$, $mnbd$, in regard to a plane through mn perpendicular to the plane $mnab$. These solids and their distances from the plane being both of opposite signs, the moments are of the same sign. Now

the sum of these two solids, being of zero mass, has the same moment about pq as about mn. The second moment of the area about pq is the first moment of the volume $aefb$ in regard to a normal plane through pq; the plane ef passing through pq. The difference, therefore, between the two second moments is the moment of $cefd$ in regard to the normal plane through pq. Now $cefd$ is equivalent to a cylinder standing on ab, of height ce or df. This height is the moment about pq of a unit of area collected at g. Hence the moment of $cefd$ in regard to the normal plane through pq is the first moment about pq of the first moment of the area supposed to be collected at g, that is, it is the second moment of the area collected at g.



Otherwise thus: let h be the distance between the two parallel lines, x the distance of any point from mn, then $h+x$ is its distance from pq. Hence second moment of $pq = \int (x+h)^2 d\alpha = \int x^2 d\alpha + 2h \int x d\alpha + h^2 \int d\alpha$. Now $\int x d\alpha$ is the first moment of the area in regard to mn, which is zero because mn passes through the mass-centre. And $\int d\alpha$ is simply the whole area α. Therefore second moment of $pq = \int x^2 d\alpha + h^2\alpha =$ second moment of mn + second moment of whole area supposed to be collected at g.

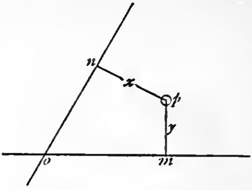
The second moment of an area in regard to any line, divided by the area itself, is the square of a length which is called the *swing-radius* of the area in regard to the line, or of the line in regard to the area.

Thus if k be the swing-radius of pq, and a that of mn, the second moment of the former is $k^2\alpha$ and of the latter $a^2\alpha$; and we have $k^2 = a^2 + h^2$.

CONJUGATE AXES. POLE OF GIVEN AXIS.

If one line pass through the pole of a second in regard to any area, the second passes through the pole of the first. Let x be the distance of a point p in the area from the

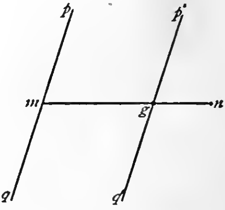
first line, and y from the second. If the area be loaded from the second line, the density at p is proportional to y; for shortness suppose it is y. Then the moment in regard to the first line is approximately $xy\delta x$. Thus the moment in regard to the first line of the area loaded from the second is $\int xydx$. We shall call this the *mixed moment* of the area in regard to the two lines. It is clearly the same as the moment in regard to the second line of the area loaded from the first. If the first line pass through the pole of the second, this mixed moment vanishes, and consequently the second line passes through the pole of the first. Two such lines are called *conjugate*.



When a line passes through the mass-centre, the area loaded from it is of zero mass; all the lines conjugate to it are parallel, and the pole is away at an infinite distance.

We may now shew how to find the pole of a given line, when the swing-radii of all lines through the mass-centre are known*. Let pq be any line, $p'q'$ the parallel through the mass-centre g, mgn conjugate to $p'q'$ and therefore to pq. Let k be the swing-radius of pq *measured parallel to mn*; that is to say, let k be a line parallel to mn whose component perpendicular to pq is the swing-radius of pq. Let a be the swing-radius of $p'q'$, measured in the same way; and let $mg = h$. Then we know that

$$k^2 = a^2 + h^2.$$



Also the distance from pq of its pole, *measured parallel to mn* is the ratio of $k^2\alpha$ to hx. Let n be the pole of pq, then

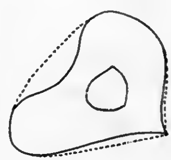
$$mn = \frac{k^2}{h} = \frac{a^2 + h^2}{h} = h + \frac{a^2}{h};$$

* [It has been pointed out to me that we must know also how to find the 'conjugate'. Culmann considers the core as the antipolar reciprocal of the contour with regard to the momental ellipse.]

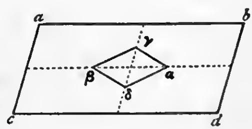
but mg being h, we see that $gn = a^2 : h$, or $mg \,.\, gn = a^2$. It follows that if a line move parallel to itself, the nearer it is to the mass-centre the further away is the pole, and *vice versâ*.

CORE OF AN AREA.

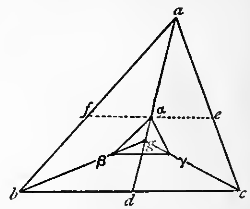
Suppose any area to have a tight string drawn round it, so that the string is everywhere either straight or convex to the outside. Let a line move round touching the string at the convex parts, coinciding with it when it is straight, and turning through the necessary angle at every sharp point. As this line moves, its pole with regard to the area will trace out a curve. Any line cutting the area will have its pole outside this curve; for it will be nearer to the mass-centre than the parallel line whose pole is on the curve. Any line not meeting the area at all will have its pole *inside* the curve. This region, containing the poles of all lines which do not cut the boundary of the area, is called the *core* of the area.



If the area is a *parallelogram*, we know that the pole of ac is at α, on the line joining the middles of ac and bd, two-thirds way to bd. Similarly the poles of cd, db, ba are at γ, β, δ. Now let a line, coinciding at first with ac, turn round c until it coincides with cd. Since $\alpha\gamma$ passes through the poles of ac and cd, both ac and cd pass through the pole of $\alpha\gamma$, which is therefore c. As the line turns round c therefore, from the position ac to cd, its pole will move along $\alpha\gamma$ from α to γ. As the line turns round d, the pole will travel from γ to β, and so on. Hence $\alpha\gamma\beta\delta$ is the core of the parallelogram. Or, *the core of a parallelogram is another parallelogram, whose diagonals are the middle thirds of the median lines.*



In the case of a *triangle abc*, the pole of bc is the mass-centre of the wedge made by tilting the triangle slightly about it. Now the mass-centre of a tetrahedron is half-way between the middles of opposite edges; consequently the pole of bc is at α the bisection of ad. The length $g\alpha = \frac{1}{4}ga$, because



$$\tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{4} \cdot \tfrac{2}{3}.$$

Similarly the poles of ca, ab are at β, γ, so that $g\beta = \frac{1}{4}gb$, $g\gamma = \frac{1}{4}gc$. Thus the core of abc is $\alpha\beta\gamma$, a similar triangle, having the same mass-centre, and of one-fourth the linear dimensions; which may be called its *middle quarter*.

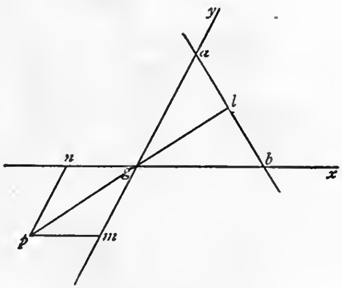
To determine the core of a *circle*, we observe first that the sum of the second moments of any area about two lines at right angles through any point is independent of the direction of the lines, and depends only on the point. For if x and y be the distances of any small portion $\delta\alpha$ of the area from the two lines, $x^2\delta\alpha + y^2\delta a = r^2\delta\alpha$, where r is the distance from the point of intersection of the lines. Thus $\int x^2 d\alpha + \int y^2 d\alpha = \int r^2 d\alpha$. It follows of course that the sum of the squares of the swing-radii is constant. The length whose square is equal to this sum may be called the *swing-radius in respect of the point* of intersection of the two lines.

In the case of the circle, then, the second moment in regard to any diameter being clearly the same is half the sum of the second moments in regard to two of them which are at right angles, or $\frac{1}{2}\int r^2 d\alpha$ where r is the distance from the centre. Suppose the circle divided into thin concentric rings of equal breadth δr, then the area of one of them will be $2\pi r\delta r$, and the second moment in regard to a diameter is $\frac{1}{2}\int_0^a 2\pi r^3 dr, = \frac{1}{4}\pi a^4$ where a is the radius of the circle. Since the area is πa^2, we see that the squared swing-radius is $\frac{1}{4}a^2$, or the swing-radius is $\frac{1}{2}a$. It follows that the core is a concentric circle of one-

quarter the radius; which we may call the *middle quarter*, as in the case of the triangle.

If we deform an area by parallel projection or homogeneous strain, it is easy to see that the core of the area will be strained into the core of the strained area. Thus it appears that the core of an ellipse is a similar, similarly situated, and concentric ellipse, of one-quarter the linear dimensions; again, as we may say, the middle quarter.

When the core is known, it is easy to construct the pole of any line, and thence to find its second moment.



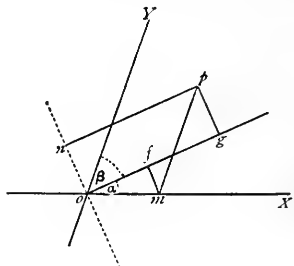
Let gx, gy be two conjugate lines through the mass-centre, ab any other line. To find the pole of ab, take m on ag so that $ag \, . \, gm =$ square of swing-radius of gx measured parallel to gy, and n on bg so that $bg \, . \, gn =$ square of swing-radius of gy measured parallel to gx. Then mp, parallel to gx, is the neutral axis whose pole is a, and np, parallel to gy, is that whose pole is b. Hence p is the pole of ab.

All we require, therefore, is to know the poles of two lines parallel to a pair of conjugate lines through the mass-centre.

If pg meet ab in l, the squared swing-radius of ab, measured parallel to gl, is $lg \, . \, lp$. Hence the second moment is known.

SWING-CONIC.

Let oX, oY be conjugate lines through any point o, and let on be any other line, the normal ofg to which makes angles α, β with oX, oY. The perpendicular dis-



tance pn of any point p from on is equal to the projection on og of op, or, what is the same thing, to the sum of the projections of om and mp on og. Let $om = x$, $mp = y$, then these projections are $x\cos\alpha$, $y\cos\beta$. Hence

$$pn = x\cos\alpha + y\cos\beta.$$

It follows that the second moment of the area in regard to on is

$$\int (x\cos\alpha + y\cos\beta)^2\, d\alpha, \quad = \cos^2\alpha \int x^2 d\alpha + \cos^2\beta \int y^2\, d\alpha,$$

since $\int xy d\alpha = 0$ because the lines oX, oY are conjugate.

Let $$\int x^2 d\alpha = a^2\alpha,\ \int y^2 d\alpha = b^2\alpha,$$

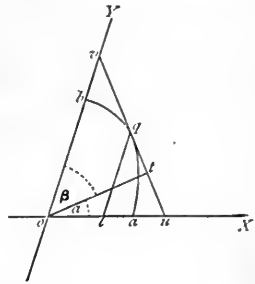
so that a, b are swing-radii of oY, oX measured parallel to oX, oY respectively. Then if k be the swing-radius of on,

$$k^2 = a^2\cos^2\alpha + b^2\cos^2\beta.$$

Now if an ellipse be drawn having o for centre and a, b for conjugate semi-diameters along oX, oY, the tangent to this ellipse parallel to on will be at a distance p from it, such that $$p^2 = a^2\cos^2\alpha + b^2\cos^2\beta;$$

so that $k = p$.

For let vqu be this tangent, $ot = p$ the perpendicular on it from the centre, ql parallel to oY; and let $ol = x$, $lq = y$. Then



$$p = ot = ou \cos\alpha = ov \cos\beta.$$

And

$$ol : oa = oa : ou = oa \cos\alpha : p,$$

or $$\frac{x}{a} = \frac{a \cos\alpha}{p};$$

so also $$\frac{y}{b} = \frac{b \cos\beta}{p}.$$

But we know that $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

therefore $$\frac{a^2 \cos^2\alpha + b^2 \cos^2\beta}{p^2} = 1,$$

or $$a^2 \cos^2\alpha + b^2 \cos^2\beta = p^2.$$

In the case of a hyperbola, it will still be true that

$$ol : oa = oa : ou,$$

because, the hyperbola being central projection of a circle, $a'lau$ is a harmonic range [p. 42]. Hence as before

$$\frac{x}{a} = \frac{a \cos\alpha}{p}, \quad \frac{y}{b} = \frac{b \cos\beta}{p};$$

and the equation [foot of p. 89]

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

gives in this case

$$p^2 = a^2 \cos^2 \alpha - b^2 \cos^2 \beta.$$

Thus, *if a pair of parallels be drawn to every line through o, at distances equal to its swing-radius on either side of it, these parallels will all touch an ellipse.* This ellipse is called the *swing-conic* at the point o. The swing-conic *at the mass-centre* may be called simply the swing-conic of the area.

The curve will always be an ellipse in those cases which occur in practice, because the area being entirely positive, $\int x^2 d\alpha$, $\int y^2 d\alpha$ must be also both positive. But if we consider also ideal areas, parts of which may be negative, the quantities $\int x^2 d\alpha$, $\int y^2 d\alpha$ may be of different signs, and then the swing-conic will be a hyperbola.

When the swing-conic at the mass-centre is a hyperbola, the second moment of every line touching the conjugate hyperbola will be zero. Suppose that $\int x^2 d\alpha = a^2$ and is positive, $\int y^2 d\alpha = - b^2$ and is negative. Then

$$k^2 = a^2 \cos^2 \alpha - b^2 \cos^2 \beta.$$

The squared swing-radius of a parallel line at distance p is

$$p^2 + k^2 = p^2 + a^2 \cos \alpha - b^2 \cos^2 \beta.$$

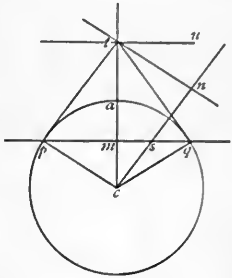
This is zero if $\quad p^2 = b^2 \cos^2 \beta - a^2 \cos^2 \alpha,$

which is the case for tangents of the conjugate hyperbola. In such a case the swing-radius k is imaginary, since its square is negative. This conjugate hyperbola is called the *null-conic* of the area.

POLES AND POLARS.

A neutral axis and its pole in regard to an area are related to the null-conic of the area in a very simple way. The geometrical properties of this relation are most easily derived from the corresponding theory in the case of the circle.

Let the tangents to a circle at p, q meet in t and let ct meet pq in m. Then we know that $cm \,.\, ct = ca^2$. The point t is called the *pole* of pq in regard to the circle, and pq is called the *polar* of t. If a line tu be drawn through t perpendicular to ct, then tu is the polar of m, and m is the pole of tu. So that a point and a line are pole and polar in regard to a circle when the product of their distances from the centre is equal to the square of the radius, and when the line is perpendicular to that radius of the circle which passes through the point. When the point is outside the circle, its polar passes through the points of contact of tangents from it to the circle.



If a point s *lies on the polar of* t, *then* t *lies on the polar of* s. Draw tn perpendicular to cs; then the triangles csm, ctn are similar, and $cm : cs = cn : ct$. Therefore

$$cs \,.\, cn = cm \,.\, ct = ca^2.$$

Hence s is the pole of tn, or t lies on the polar of s.

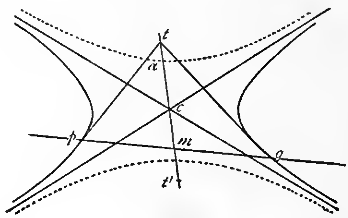
In this case the points s, t are called *conjugate points*, and their polars are called *conjugate lines* in regard to the circle.

It follows that if through any point s we draw a series of chords pq to meet the circle, the tangents at the extremities of these chords will all meet on a certain straight line, which is the polar of s. For since each of these chords passes through s, its pole must lie on the polar of s.

Now this last property of the circle is one which must belong to all central projections of the circle. For the projections of the chords pq will be a series of straight lines passing through the projection of s; and the tangents at their extremities will all meet on the projection of tn, which we know to be a straight line. Hence the property is true also of the ellipse, parabola, and hyperbola.

In the case of the ellipse, moreover, it remains true that if ct meets the polar of t in m, and the ellipse in a, then

$$cm \,.\, ct = ca^2.$$



But in the case of the hyperbola, when the polar meets both branches of the curve, this statement requires interpretation. In that case, as the figure shews, ct does not meet the hyperbola itself; but if it meets the conjugate hyperbola in a, we shall still have

$$cm : ac = ca : ct, \text{ or } cm \,.\, ct = -ca^2.$$

Hence in order to apply the theorem in this case, we must regard the distance from the centre to the original hyperbola, measured along ct, as being the square root of $-ca^2$, or $ca\sqrt{(-1)}$. This $\sqrt{(-1)}$ however is not the operation of turning through a right angle; for if we turn ca through a right angle, we no longer get a distance measured along ct. It must be treated for the present simply as a means of simplifying the statement of certain propositions. To determine the points where ct meets the hyperbola, we must measure off on either side of c the distance $ca\sqrt{(-1)}$. Let a_1, a_1' be the points so found; we say that these points are *invisible;* but still, since $ca_1^2 = -ca^2$, it follows that $cm \,.\, ct = ca_1^2$.

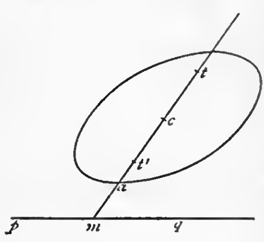
In this way we shall take the squared semi-axes of the hyperbola to be a^2, $-b^2$; and the squared semi-axes of the conjugate hyperbola will be $-a^2$, b^2. It will be seen at once that by this consideration many formulæ relating to the hyperbola become reconciled with the corresponding formulæ for the ellipse. For example, in both curves the

squared distance of the foci from the centre is equal to the *difference* of the squares of the semi-axes.

The pole of pq in regard to the conjugate hyperbola is a point t' such that $cm . ct' = ca^2$. Thus t, t' are *opposite points* in regard to the centre c. The conjugate hyperbola has its squared semi-axes equal in magnitude but opposite in sign to those of the original hyperbola. Thus when two conics have their axes equal in magnitude and opposite in sign, the poles of the same line in regard to them are opposite points in regard to their common centre.

APPLICATION TO THE NULL-CONIC.

We may shew very easily that the pole of any line in regard to an area is the opposite point of its pole in regard to the swing-conic at the mass-centre. Let pq be the line, c the mass-centre, cam conjugate to a line through c parallel to pq. We have seen that the swing-radius of this parallel line is equal to the perpendicular from c on the tangent at a to the swing-conic; for two lines conjugate in regard to the area are conjugate diameters of the swing-conic at their intersection, so that the tangent at a will be parallel to pq. Hence the swing-radius of the parallel diameter, measured parallel to ca, will be ca itself. Therefore the pole of pq in regard to the area is at a point t such that



$$mc . ct = ca^2.$$

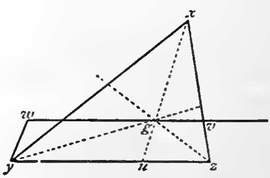
Now its pole in regard to the swing-conic is a point t' such that $cm . ct' = ca^2$. Therefore $tc = ct'$, or t, t' are opposite points in regard to c.

If therefore the swing-conic is a hyperbola, the pole of a line in regard to the area is simply its geometric pole in regard to the null-conic. But also when the swing-conic

is an ellipse, there is a certain convenience in saying the same thing. For since $cm \cdot ct = -ca^2$, if we write $ca_1^2 = -ca^2$, or $ca_1 = ca\sqrt{(-1)}$, we shall have $cm \cdot ct = ca_1^2$. Hence if we construct a conic having the same centre and direction of axes as the swing-conic, but having for squared semi-axes $-a^2$ and $-b^2$ instead of a^2 and b^2, so that every diameter of it is equal to the corresponding diameter of the swing-conic multiplied by $\sqrt{(-1)}$, then the pole of any line in regard to the area is its geometric pole in regard to this conic. The conic of course is altogether invisible; yet it gives rise to a visible and perfectly definite system of poles and polars, so that we can find the pole of any given line or the polar of any given point in regard to it. The only difference is that whereas in regard to an ellipse the pole and polar are always on the same side of the centre, and in regard to a hyperbola they are sometimes on the same and sometimes on opposite sides, according as the polar does not or does cut both branches; in regard to this invisible conic the pole and polar are always on opposite sides of the centre. If we agree to take the notion of a conic as involving not only the points and tangents of the curve but also the system of poles and polars to which it gives rise, we may say that there always *is* a null-conic, but that when the area is all positive its points and tangents are invisible; and that the pole of any line in regard to the area is its geometric pole in regard to the null-conic.

EQUIVALENT TRIAD OF PARTICLES.

We shall now prove that for purposes of calculating second moments, an area may be replaced by three particles. Take any point x in the plane of the area; let y be a point on its neutral axis (or *polar*, as we may now call it), and let z be the pole of xy. Then xyz is a triangle of which every vertex is the pole of the opposite side; such a triangle is called *self-*



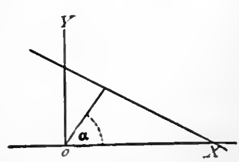
*conjugate**. Let xg meet yz in u, then $xg \cdot gu$ is the square of the semi-diameter along xu of the swing-conic at the mass-centre g. If a line through g parallel to yz meet xz in v and yw, parallel to gx, in w, then v is the pole of yw and consequently $wg \cdot gv$ is the square of the semi-diameter along wv. Thus if x, y, z and g are known, we can find the lengths and position of two conjugate semi-diameters of the swing-conic; that is, the swing-conic is determined.

Now when we know the mass α of an area and the swing-conic at the mass-centre, the second moment in regard to every line in its plane is determined. For it is $\alpha(h^2 + a^2)\sin^2\theta$, where a is the conjugate semi-diameter, h the distance from the centre measured along it, and θ the angle it makes with the given line.

If three particles be placed at xyz, their masses being in such proportion that their mass-centre is g, they will constitute a system (regarded as an area shrunk up into three points) which obviously has xyz for a self-conjugate triangle, and therefore the same swing-conic at g as the given area. If we now make the sum of their masses $= \alpha$, their second moment about any line in the plane is the same as that of the given area. Thus *so far as second moments are concerned, an area may be replaced by three particles, forming a self-conjugate triangle, of the same resultant mass as the given area.*

PRINCIPAL AXES.

The axes of the swing-conic at any point are called the *principal axes* of the area at that point. The principal axes at the mass-centre are often called for shortness principal axes of the area.



Let oX, oY be the principal axes at the mass-centre o, and let b, a be their swing-radii. Then the squared swing-radius of a line, the perpendicular from the origin on which is of

* [Self-conjugate with regard to the area, it is not self-conjugate with regard to swing-conic.]

length p and inclination α to the axis oX, is

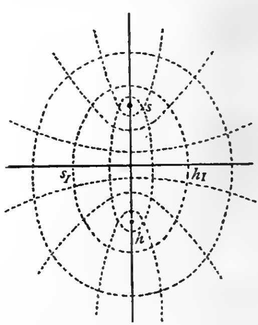
$$k^2 = a^2 \cos^2\alpha + b^2 \sin^2\alpha + p^2.$$

Consequently $\quad p^2 = (k^2 - a^2)\cos^2\alpha + (k^2 - b^2)\sin^2\alpha;$

whereby it appears that this line touches a conic the squared semi-axes of which are $k^2 - a^2$, $k^2 - b^2$.

Suppose that a is greater than b; then the swing-conic is an ellipse whose foci are on oX at distances $\sqrt{(a^2 - b^2)}$ from o on either side. The conic whose squared semi-axes are $k^2 - a^2$, $k^2 - b^2$ has its foci on oY at the same distances from o. Since the squared semi-axes of the null-conic are $-a^2$, $-b^2$, it also has its foci on oY at this distance. Hence *all the lines whose swing-radius is equal to a given quantity k touch a conic confocal with the null-conic.*

Let s_1, h_1 be foci of the swing-conic, s, h of the null-conic and of all the conics of constant swing-radius. When k is greater than a, the conic is an ellipse; when it is between a and b, a hyperbola; when less than b, invisible. Through every point p pass one ellipse of the series, for which the sum of the focal distances is $sp + hp$; and one hyperbola, for which their difference is $sp \sim hp$. Hence every ellipse cuts every hyperbola, but no two ellipses or hyperbolæ intersect. The curves cut at right angles, because their respective tangents bisect internally and externally the angle sph. To pass from any ellipse to a larger one is to increase the value of k.



The principal axes at any point are tangents to the ellipse and hyperbola confocal to the null-conic which pass through that point. For clearly one principal axis has the greatest swing-radius, and the other the least, of all axes passing through the point. Now consider the tangent to an ellipse of the system at a point p. Its swing-radius is

the quantity k belonging to that ellipse. If we turn the line round p a little, it will cut the ellipse; thus the curve of the series which it touches is an ellipse *inside* the given one, which will therefore belong to a less value of k. Thus by turning the tangent round, we diminish its swing-radius; or the tangent to the ellipse is the line of greatest swing-radius through p; that is, it is one of the principal axes. Similarly the tangent to the hyperbola is the line of least swing-radius, and therefore the other principal axis.

SECOND MOMENTS OF A SOLID.

The preceding theory may now be very readily extended to three dimensions.

If a body be altered by multiplying its density at every point by the distance from a given plane, it is said to be *loaded from* that plane. Distances on one side of the plane being reckoned positive, those on the other side are reckoned negative; thus the density of the body is altered in sign on one side of the plane.

The first moment of the loaded body in regard to the plane is called the *second moment* of the unloaded body in regard to the plane. If the body be divided into particles, the mass of one of which is δm, the second moment is what $\Sigma x^2 \delta m$ indefinitely approximates to with diminishing δm; that is, it is $\int x^2 dm$; where x is the distance of δm from the plane.

The mass-centre of the loaded body is called the *pole* of the plane in regard to the unloaded body.

If one plane pass through the pole of another, the second plane passes through the pole of the first. For in that case $\int xy\,dm = 0$, where x, y are distances from the two planes. $\int xy\,dm$ is the *mixed moment* in regard to the two planes, which are called *conjugate* when it vanishes.

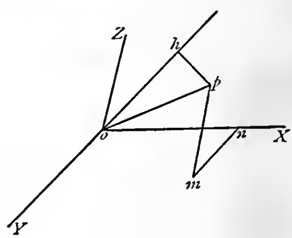
The quotient of the second moment of a body in regard to any plane by the mass of the body, or $\int x^2 dm : m$, is the square of a length k which is called the swing-

radius of the body in regard to that plane. Thus $\int x^2 dm = k^2 m$. If the distances x are drawn perpendicular to the plane, the quantity k is called the swing-radius simply; but if they are measured in any other direction, the ratio $\int x^2 dm : m$ is the square of the swing-radius *measured in that direction*. Thus the *swing-radius measured in any direction* is a line in that direction whose component normal to the plane is the swing-radius k.

SWING-ELLIPSOID.

Take any plane through a point o, then a second plane through o and the pole of the first, then a third through o and the poles of the other two. Then each of these planes is conjugate to the other two.

Let oX, oY, oZ be their lines of intersection, and oh be a line making angles $\alpha\beta\gamma$ with these lines respectively. Then the distance from a point p to a plane Q through o perpendicular to oh is equal to ho, if ph is perpendicular on oh. Now this is the projection on oh of op, and therefore it is the sum of the projections of on, nm, mp, the coordinates of p, which shall be called x, y, z. These projections are $x \cos \alpha$, $y \cos \beta$, $z \cos \gamma$; and so



$$ho = x \cos \alpha + y \cos \beta + z \cos \gamma.$$

Thus the second moment of the body in regard to the plane Q, through o perpendicular to oh, is

$$\int (x \cos \alpha + y \cos \beta + z \cos \gamma)^2 dm, \ = \cos^2 \alpha \int x^2 dm + \cos^2 \beta \int y^2 \, dm + \cos^2 \gamma \int z^2 dm,$$

since $$\int yzdm = \int zxdm = \int xydm = 0,$$

because each coordinate plane passes through the poles of the other two. Let

$$\int x^2 dm = ma^2, \ \int y^2 dm = mb^2, \ \int z^2 dm = mc^2,$$

where m is the mass of the body, so that a, b, c are *swing-radii* of the three planes, measured respectively parallel to the three lines of intersection; and let k be the swing-radius of the plane Q. Then

$$k^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma.$$

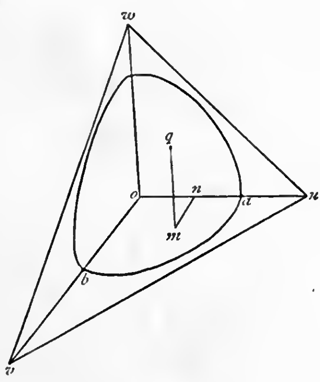
Now let an ellipsoid be constructed with semi-conjugate diameters a, b, c along oX, oY, oZ; then it may be shewn that k is the perpendicular on the tangent plane of this ellipsoid parallel to the plane Q.

For let uvw be tangent plane at a point q; then the perpendicular p on it from o is equal to

$$ou \cos \alpha = ov \cos \beta = ow \cos \gamma,$$

if α, β, γ are the angles which that perpendicular makes with the axes. And $on : oa = oa : ou = oa \cos \alpha : p$;

or
$$\frac{x}{a} = \frac{a \cos \alpha}{p}.$$



Now
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

therefore
$$a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma = p^2.$$

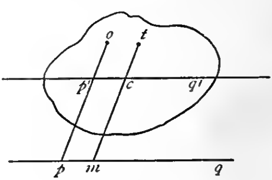
Hence if parallel planes be drawn on either side of every plane through o at distances equal to the swing-

radius of that plane, they will all touch an ellipsoid; which is called the swing-ellipsoid at the point o.

If part of the mass of the body is negative, so that the quantities $\int x^2dm$, $\int y^2dm$, $\int z^2dm$ may not all be positive, the pairs of parallel planes may touch a hyperboloid; so that if we take into account the possibility of negative mass we must speak of the swing-*quadric*, and not of the swing-ellipsoid. The results which depend upon negative mass are however not susceptible of the same interpretations in solid bodies as in the case of areas.

DETERMINATION OF THE POLE OF ANY PLANE.

The squared swing-radius of any plane is equal to that of the parallel plane through the mass-centre together with the square of the distance between them; all being measured in any given direction. The proof is the same as in two dimensions. Let h be the distance pp' between the parallel planes pq and $p'q'$, and let x be the distance of any point o from the plane $p'q'$ which goes through the mass-centre. Then the second moment of



$$pq = \int (x+h)^2 dm = \int x^2 dm + 2h\int x dm + h^2 \int dm = \int x^2 dm + h^2 m,$$

because, $p'q'$ passing through the mass-centre, its first moment is zero, or $\int x dm = 0$. Hence if k be the swing-radius of pq and a of $p'q'$, we have $k^2 = a^2 + h^2$.

The pole t of the plane pq is on the line mc through the mass-centre conjugate to $p'q'$, in such a position that $mc \,.\, ct = a^2$, the squared swing-radius of $p'q'$ measured parallel to mc. For we know that mt multiplied by the first moment of pq is equal to the second moment. Therefore

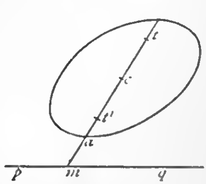
$$mt = \frac{a^2+h^2}{h} = h + \frac{a^2}{h} = mc + ct, \text{ or } mc \,.\, ct = a^2.$$

RELATION OF POLE TO SWING-QUADRIC.

The theory of *poles* and *polar planes* in regard to quadric surfaces is precisely analogous to that of poles and polar lines in regard to conics. From a point external to a sphere we can draw an infinite number of tangent lines to the sphere, which altogether form a cone touching the sphere and having its vertex at the point. The points of contact of all these tangent lines are in one plane, called the *polar plane* of the point, which is perpendicular to the line joining its pole to the centre, and such that the product of the distances from the centre of the pole and its polar plane is equal to the square of the radius. The truth of these statements can be seen by supposing the figure on p. 25 to be spun round the line ct, when the circle will trace out a sphere, the lines tp, tq the tangent cone, and pq the polar plane of t. From the same figure it may be seen that if one point lies on the polar plane of a second, the second point will lie on the polar plane of the first. Hence if a series of planes be drawn through any point inside or outside of a sphere, and cones be drawn touching the sphere along the circles where these planes meet it, the vertices of all these cones will lie in a plane, called the polar plane of the point.

The last theorem may now be extended to the ellipsoid by a homogeneous strain, and to other quadric surfaces by first proving it for surfaces of revolution, as with the theorems on p. 25.

This being so, we see at once that if we draw the swing-ellipsoid at the mass-centre, the pole of any plane in regard to the body will be the opposite point of its pole in regard to this ellipsoid. For if a be the point of contact of a tangent plane parallel to pq, ca will be the swing-radius measured in that direction of a plane through c parallel to pq; and



therefore the pole t of pq in regard to the body is so

situated that $mc \,.\, ct' = ca^2$. Now the pole t' in regard to the ellipsoid is so situated that $cm \,.\, ct' = ca^2$. Hence t, t' are opposite points in regard to c.

The point t may also be taken to be the pole of pq in regard to an invisible quadric surface whose semi-diameter in the direction cm is $ca\sqrt{-1}$; because $cm \,.\, ct = -ca^2$. This invisible surface is called the *null-quadric*, because the second moment of its tangent planes is zero. In fact, the second moment of a plane parallel to pq at a distance h from c measured parallel to cm is $m(ca^2 + h^2)$. Hence if $h^2 = -ca^2$, or $h = ca\sqrt{-1}$, this is zero.

EQUIVALENT TETRAD OF PARTICLES.

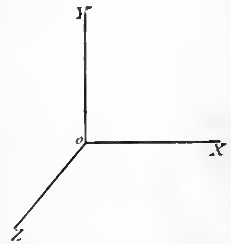
Just as a plane area may be replaced by three particles, so far as second moments are concerned, so a solid body may be replaced by a system of four particles. These must be so placed that each is the pole of the plane joining the other three, and their masses must be equivalent to the mass of the body, i.e., they must have the same mass-centre as the body, and the sum of their masses must be m. Four points such that each is the pole of the plane joining the other three in regard to a quadric surface, are said to form a *self-conjugate tetrahedron* of that surface. The theorem is therefore that any body is equivalent, so far as second moments are concerned, to four particles, of the same resultant mass as the body, placed at the vertices of a self-conjugate tetrahedron of the null-quadric.

Let x, y, z, w be the four points; then, in regard both to the body and to the system of four particles, the swing-quadric at x has the lines xy, xz, xw for a set of semi-conjugate diameters. For clearly the line xy from x to the pole of xzw is the swing-radius of that plane measured in the direction xy. Hence the two systems have the same swing-conic at x. But the squared swing-radius of any plane through the mass-centre is got from the squared swing-radius of a parallel plane through x by subtracting the squared distance between them. Hence the two

systems have the same swing-conic at their common mass-centre; so that, since they have the same mass, their second moments in regard to any plane are equal.

SECOND MOMENTS IN REGARD TO AN AXIS.

If two planes at right angles to one another be drawn through any straight line, the sum of their second moments in regard to any body is called the second moment of the body about that line as axis.



Let oZ be the line, oZX, oZY the two perpendicular planes, oX and oY lines in them at right angles to oZ; and let x, y, z be coordinates of a point p in regard to this system of coordinate planes. If the body is divided into particles, one of which having the mass δm contains the point p, then the second moments of the body in regard to the two planes are $\int x^2 dm$ and $\int y^2 dm$. Thus the sum of them is $\int (x^2 + y^2)\, dm$. If r is the perpendicular from p on the axis oZ, we know that $r^2 = x^2 + y^2$; thus the second moment of the body about oZ is $\int r^2 dm$. Hence, *to find the second moment of a body in regard to an axis, divide the body into particles, and multiply the mass of each particle by the square of its distance from the axis; the sum of all these products will approximate as nearly as we like to the second moment if we take the particles small enough.* It follows that the second moment about an axis is independent of the pair of perpendicular planes that we take.

The quotient of this second moment by the mass of the body is the square of a length which is called the swing-radius about the given axis.

We may shew in the same way that the sum of the second moments of three perpendicular planes through a fixed point depends only on the point, and not on the aspects of the three planes. For it is $\int (x^2 + y^2 + z^2)\, dm$;

now $x^2+y^2+z^2$ is the squared distance of the point p from the origin o, so that this quantity may be reckoned by multiplying the mass of each particle by its squared distance from the origin. It may be called the second moment of the body about the point o.

It is immediately obvious that the sum of the second moments of a body in regard to an axis and a plane which cut at right angles is equal to the second moment of the body about their point of intersection.

The squared swing-radius about any axis is equal to that about a parallel axis through the mass-centre together with the squared distance between them. For let A, B be two perpendicular planes through the axis, of which A passes through the mass-centre; and let B' be a plane through the mass-centre parallel to B. Then the difference of the second moments of the axes AB and AB' will be the difference of the second moments of the planes B, B', which we know to be the mass of the body multiplied by the squared distance between them.

In a similar manner it may be shewn that the second moment of a body about any point is equal to that about the mass-centre together with the product of the mass of the body by the squared distance of the point from the mass-centre.

ELLIPSOID OF GYRATION.

Let oX, oY, oZ be the directions, and a, b, c the magnitudes, of the semi-axes of the swing-ellipsoid at any point o of the body; then the squared swing-radius of a plane whose normal makes angles α, β, γ with these axes is $a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma$. We shall get the second moment of the body about an axis through o normal to this plane if we subtract the second moment of the plane from the second moment of the point o. This latter quantity is $m(a^2+b^2+c^2)$, which if we remember that

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,$$

may be written

$$m(a^2+b^2+c^2)(\cos^2\alpha+\cos^2\beta+\cos^2\gamma).$$

Hence the squared swing-radius of the axis which makes angles $\alpha\beta\gamma$ with oX, oY, oZ is

$$k^2 = (a^2 + b^2 + c^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$
$$\qquad - (a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma)$$
$$= (b^2 + c^2)\cos^2\alpha + (c^2 + a^2)\cos^2\beta + (a^2 + b^2)\cos^2\gamma.$$

Here the quantities $b^2 + c^2$, $c^2 + a^2$, $a^2 + b^2$ are squared swing-radii about the axes oX, oY, oZ. We shall denote them by the letters A, B, C, so that

$$k^2 = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma.$$

From this formula it appears that if we construct an ellipsoid having A, B, C for its squared semi-axes along oX, oY, oZ, the swing-radius about any axis through o will be equal to the length of it cut off by the perpendicular tangent plane to this ellipsoid. The ellipsoid so constructed is called the *ellipsoid of gyration* at the point o.

CONFOCAL SURFACES.

We have seen that when different conics have a common centre and axes in the same direction, they will also have the same foci when the difference of the squares of their axes is the same. So that a conic whose squared semi-axes are a^2, b^2 will be confocal to a conic whose squared semi-axes are $a^2 + \lambda$, $b^2 + \lambda$, where λ is any arbitrary quantity.

By analogy, two quadric surfaces are said to be *confocal* when they have the same centre and their axes on the same straight lines, and when moreover the differences of the squares of their axes are the same. Thus if one surface has squared semi-axes a^2, b^2, c^2, and another

$$a^2 + \lambda,\quad b^2 + \lambda,\quad c^2 + \lambda,$$

these surfaces are said to be confocal. The conics in which they are cut by any of the principal planes are in fact confocal in the sense already considered.

If we begin with an ellipsoid of semi-axes a, b, c, and gradually increase these so as to keep the differences of their squares constant, we shall obtain a series of continually increasing ellipsoids, each entirely outside the preceding ones. These will approach to a spherical form as they get larger, because the ratios

$$a^2 + \lambda : b^2 + \lambda : c^2 + \lambda$$

approach to unity when λ is made very large. The series tends therefore as a limit to an infinitely large sphere.

If we *diminish* the axes of the ellipsoid, still keeping the differences of their squares constant, we shall obtain a series of decreasing ellipsoids, each entirely contained in the preceding ones. These will get flatter and flatter as λ approaches the value $-c^2$ (if c is the *least* of the semi-axes); and then the surface takes the form of a flat ellipse in the plane of a, b, whose squared semi-axes are $a^2 - c^2$, $b^2 - c^2$. This is called the *focal ellipse* of the whole series of surfaces.

We may now go on to give to λ a negative value greater than c^2, so that the squared axis $c^2 + \lambda$ becomes negative. This indicates that the surface has become a hyperboloid of one sheet, which is met in visible points by two of its axes, but not by the third. This surface may be regarded as starting from a flat plate, consisting of that portion of the plane a, b which is outside of the focal ellipse. It cuts the plane ab in an ellipse confocal with that one, and *inside* it; because its axes go on continually diminishing. This ellipse at last shrinks into the line joining its two foci; this is when $b^2 + \lambda$ vanishes, by λ becoming equal to $-b^2$, (if b is the *mean* semi-axis, and a the greatest). The one-sheeted hyperboloid has then become a flat plate in the plane a, c, consisting of that portion of the plane which is between the two branches of the hyperbola whose squared semi-axes are

$$a^2 - b^2, \quad c^2 - b^2.$$

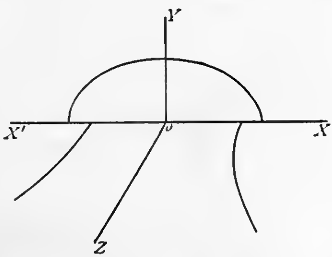
This is called the *focal hyperbola* of the system of surfaces.

Giving now to λ a series of values negatively greater than b, we shall obtain a series of two-sheeted hyperboloids, which start with the remaining portion of the plane a, c, namely that which is inside each branch of the focal hyperbola. They cut that plane in hyperbolæ confocal to it, and lying between its branches. The two sheets of these hyperboloids approach one another and the plane b, c, as λ approaches the value $-a^2$. Then they unite into a flat plate consisting of the whole of that plane, in which the focal conic is invisible, having the squared semi-axes $b^2 - a^2$, $c^2 - a^2$.

After this, if we continue to increase the negative value of λ, the surface becomes wholly invisible, all three squared semi-axes being negative; but we must regard it as continually increasing in size until, for an infinite value of λ, it coincides with the infinite sphere before mentioned.

This discussion will have made it easy to see that each of the three series,—the ellipsoids, the one-sheeted hyperboloids, and the two-sheeted hyperboloids,—sweeps over the whole of space; so that through every point it is possible to draw one surface of each kind belonging to the system.

The relation to one another of the focal ellipse and focal hyperbola may be understood from this figure. They are in perpendicular planes, and each passes through



the foci of the other. It is found that the sum or difference of the distances from any two fixed points on one of them to a variable point on the other is constant.

In the particular case in which two of the quantities a, b, c are equal, all the surfaces of the system are surfaces of revolution; and the system is obtained by rotating the figure of p. 30 about sh if $b=c$, or the two lesser axes are equal, and about s_1h_1 if $a=b$, or the two greater axes are equal. In the former case there are two separate foci on the axis of revolution, and an invisible focal circle in the equatoreal plane. In the latter case there is a real focal circle traced out by s_1, h_1, and two invisible foci on the axis of revolution.

PRINCIPAL AXES.

We shall now prove that *the axes of the swing-ellipsoid at any point* (*principal axes* at the point) *are normals to the three surfaces confocal to the null-quadric which can be drawn through the point.*

Let a, b, c be semi-axes of the swing-ellipsoid at the mass-centre o, and let oX, oY, oZ be their directions. Then if k be the swing-radius of a plane at the distance p from o, the normal to which makes angles α, β, γ with oX, oY, oZ, we shall have $a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma+p^2=k^2$, which may be written $k^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$ because $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. Hence

$$p^2=(k^2-a^2)\cos^2\alpha+(k^2-b^2)\cos^2\beta+(k^2-c^2)\cos^2\gamma.$$

From this it follows that every plane whose swing-radius is k touches the quadric surface whose squared semi-axes are k^2-a^2, k^2-b^2, k^2-c^2. This surface is confocal to the null-quadric, whose squared semi-axes are $-a^2$, $-b^2$, $-c^2$, and to the ellipsoid of gyration, which is got by putting $k^2=a^2+b^2+c^2$. Each plane touches one surface only of the system, because no two surfaces can have the same value of k.

Now consider a point q; draw through it the ellipsoid, E, of this confocal system, and its tangent plane, P. For this plane k is greater than a, b, or c, because k^2-a^2, k^2-b^2, k^2-c^2 are all positive. Any increase of k will increase the axes of the ellipsoid, and *vice versa.* Now if

we tilt the plane a little round q in any direction, so that it cuts the ellipsoid E, the surface of the system which touches the tilted plane will be an ellipsoid wholly inside of E. Hence k will be less for the tilted plane, because the axes of the touching ellipsoid are diminished. Therefore P is the plane through q which has the greatest swing-radius. Now the plane through any point which has the greatest swing-radius, is normal to the longest axis of the swing-ellipsoid at the point. Therefore the longest axis of the swing-ellipsoid (and consequently the shortest axis of the ellipsoid of gyration) at the point q, is normal to the ellipsoid E, confocal to the null-quadric, which passes through q.

Let R be the tangent plane at q to the two-sheeted hyperboloid, H, of the system, which passes through q. This surface, like the ellipsoid, lies entirely on one side of the tangent plane in the neighbourhood of the point of contact. Hence if we tilt the plane a little it will cut the surface in a small oval curve; and therefore the surface of the system which touches the new position of the plane will be a two-sheeted hyperboloid lying further away from the centre than H, and therefore having a larger transverse axis. Consequently k is increased by the tilting; whence it follows that R is the plane of least swing-radius that can be drawn through q. This plane is normal to the shortest axis of the swing-ellipsoid, or longest axis of the ellipsoid of gyration.

We may put these results together by saying that at any point the axis of least moment is normal to the ellipsoid, and the axis of greatest moment to the two-sheeted hyperboloid, which can be drawn through the point confocal to the null-quadric.

It follows that the ellipsoids and the two-sheeted hyperboloids of the system cut each other everywhere at right angles. We shall now prove that the third principal axis is normal to the one-sheeted hyperboloid through q, so that the three systems of surfaces cut each other everywhere at right angles. [Cetera desunt.]

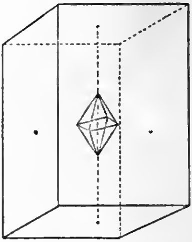
CORE OF A SOLID.

The poles of all those planes which do not cut the boundary of a solid fill up a certain space which is called its *core*. If the solid has indentations we must suppose a membrane to be tightly wrapped round it; then the boundary of the core will be traced out by the pole of a plane which moves about so as always to touch this membrane.

The main use of knowing the core of the simpler class of solids is that it is more easily remembered than a formula, and supplies a ready means of finding the swing-radius about any axis.

We have seen that the core of a finite line, which is of one dimension, is the middle third; and that the core of a triangle or ellipse, which may be regarded as typical figures of two dimensions, is the middle fourth. We shall shew that the core of a tetrahedron or ellipsoid, which are figures in three dimensions, is the middle fifth; that is, it is a figure similar to the original, having the same mass-centre, and of one-fifth of the linear dimensions.

A *parallelepiped* is to be taken as a figure of one dimension in regard to each of the lines joining the centres of opposite faces. Its core is accordingly an octahedron having the middle thirds of those lines for its three diagonals. This figure is the *reciprocal* of the parallelepiped, in the sense that the vertices, edges and faces of one of them are polar to the faces, edges, and vertices of the other. Thus the parallelepiped has six four-sided faces and eight three-legged vertices, while the octahedron has six four-legged vertices and eight three-sided faces.



For a triangular prism, or elliptic cylinder, we must as before take the middle third of the line joining the mass-centres of the opposite plane ends, and then join the ends

of this middle third to the middle fourth of a section of the solid made by a plane parallel to the two ends and midway between them. We shall thus form a core which is in one case a pentacron, like that considered on p. 8, and in the other case is made of two cones with a common base and vertices on opposite sides of it. The pentacron, having two three-legged and three four-legged vertices, with six three-sided faces, is the reciprocal of the triangular prism, which has two three-sided and three four-sided faces, with six three-legged vertices.

In general, we may obtain the core of any prism or cylinder, bounded partly by two parallel plane faces or ends, and partly by a surface made up of parallel straight lines, by taking the ends of the middle third of the line which joins the mass-centres of the plane ends, and joining them to the core of the middle section, which is parallel to the plane ends and midway between them. The figure so formed consists always of two pyramids or cones, having a common base, and vertices on either side of it.

To determine the core of a tetrahedron, it is sufficient to find the pole of a plane through one vertex a parallel to the opposite face, bcd. If the tetrahedron when loaded from this plane be divided into slices of equal thickness by planes parallel to it, the areas of the slices will vary as the squares of their distances from a, and their densities directly as those distances; therefore their masses will be proportional to the cubes of the distances from a. If g be the mass-centre of the tetrahedron, α of the face bcd, the mass-centres of all these slices will be in $a\alpha$, and consequently the tetrahedron may be replaced by a rod $a\alpha$ whose density varies as the cube of the distance from a. Thus the mass-centre of the loaded tetrahedron (which is the pole of the plane through a parallel to bcd) is $\frac{4}{5}$ of the way from a towards α, and its distance from a is therefore $\frac{4}{5} \cdot \frac{4}{3} ag = \frac{16}{15} ag$. Thus the squared swing-radius, measured parallel to ag, of a plane through g parallel to bcd is $\frac{1}{15} ag^2$. Therefore the pole of bcd, whose distance from g measured along ag is $\frac{1}{3} ag$, must be at a point whose distance is $\frac{1}{5} ga$. Now the core of the tetrahedron is clearly a tetrahedron whose vertices are the poles of

the faces; and we see that these poles are on the lines ga, gb, gc, gd, at distances from g equal to one-fifth of those lines respectively. Thus the core is the middle fifth.

For a sphere we must proceed in the same way as for a circle. The squared swing-radius of the centre in regard to the sphere is the sum of the squared swing-radii of three mutually perpendicular planes through it, and is therefore three times that of either of them. If we suppose the sphere made of concentric shells of equal thickness, the masses of these shells are as the squares of their distances from the centre; the problem is therefore the same as that of finding the squared swing-radius of a rod, whose density varies as the squared distance from one end, in regard to that end. If a is the length of the rod, the second moment is $\frac{1}{5}a^5$, and the mass is $\frac{1}{3}a^3$; thus the squared swing-radius is $\frac{3}{5}a^2$. This is therefore the squared swing-radius of a sphere of radius a in regard to its centre. In regard to a diametral plane it is one-third of this, or $\frac{1}{5}a^2$; from which it follows at once that the core is a concentric sphere of the radius $\frac{1}{5}a$.

If any body be transformed by a homogeneous strain, the core of the strained body is the strained position of the core of the unstrained body. For let x be the distance of a particle δm of the body from a given plane, measured in a given direction, and k the swing-radius in regard to that plane, measured in the same direction. Then

$$mk^2 = \int x^2 \delta m.$$

If the body receive a homogeneous strain, so that the masses of corresponding volumes are unaltered, the lines x and k, which are all parallel, will remain parallel and be altered in the same ratio; so that if every x becomes λx, k will become λk. Therefore $m \,.\, (\lambda k)^2 = \int (\lambda x)^2 \delta m$, or λk is the swing-radius measured parallel to the λx in the strained body. Hence it follows that the pole of the given plane, the swing-ellipsoid at every point, the null-quadric, and the core of the body, all undergo the same homogeneous strain as the body does. It should be noticed that this is not true of the ellipsoid of gyration.

Hence it follows, by homogeneous strain of a sphere, that the core of an ellipsoid is a similar, similarly situated, and concentric ellipsoid, of one-fifth the linear dimensions; or, as we may say, it is the middle fifth of the body.

We may exemplify the use of the core by finding the swing-radius of a sphere about certain axes. For an axis through the centre the squared swing-radius is twice that of a diametral plane, and therefore $\frac{2}{5}a^2$, since the core is the middle fifth. Hence about an axis touching the sphere the squared swing-radius is $\frac{7}{5}a^2$.

Similarly in the case of a right circular cylinder, height $2h$, radius a, the squared swing-radius of its middle section is $\frac{1}{3}h^2$, and of a plane through its axis $\frac{1}{4}a^2$. Hence about the axis it is $\frac{1}{2}a^2$, about a line through the centre perpendicular to it $\frac{1}{4}a^2 + \frac{1}{3}h^2$, and about a diameter of either plane face $\frac{1}{4}a^2 + \frac{4}{3}h^2$.

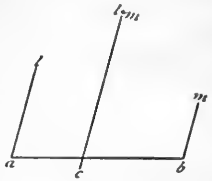
CHAPTER III. MOMENTUM.

MOMENTUM OF TRANSLATION-VELOCITY.

The product of the mass of a particle by its velocity is called the *momentum* of the particle. Like the velocity, it is a *directed* quantity; but whereas the velocity is a *vector*, and may be specified by a line of proper length and direction drawn anywhere in space, the momentum is what we have called a *rotor*, and is situated in the straight line through the particle along which it is moving. In fact, the velocity is a magnitude associated with a direction, and the mass is a magnitude associated with a definite point; thus the momentum, which is the product of these two, is the product of their magnitudes associated with a line drawn through the point in the given direction.

To compound together the momenta of two or more particles, therefore, we must treat them as if they were rotations about axes through the particles [p. 124]. If, for example, particles of masses l, m situated at a, b, have the same velocity, their momenta are proportional to the masses l, m, and may be represented by lengths l, m measured on lines through a, b in the direction of the common velocity. If the lengths represented spins about these lines as axes, the resultant spin would have a magnitude equal to their sum, and would be about a parallel line through the point c, which divides ab in the ratio

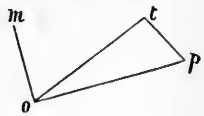
of $m : l$. Now c is the mass-centre of the particles at a, b, and its velocity is the same as that of a and b. Hence *the resultant of the momenta of two particles which have the same velocity is the momentum of their resultant mass.*



If we add to these a third particle, with the same velocity, we may compound its momentum with that of the resultant mass of the first two, and so find that the resultant of all the momenta is the momentum of the resultant mass of the three; and so on. Hence *if a body have a translation-velocity, the resultant of the momenta of its particles is equal to the momentum of its resultant mass;* that is to say, its magnitude is the mass of the whole body multiplied by the magnitude of the given velocity, and acts in a straight line through the mass-centre in the direction of the given velocity.

MOMENT OF MOMENTUM.

The *moment* of the momentum of a particle in regard to any point is defined in the same way as the moment of any similar magnitude [p. 92]. Namely, if p be the position of the particle, pt a line representing its momentum, the moment of the momentum about o is twice the area of the triangle opt. This is to be regarded as a *vector* perpendicular to the plane opt, but *not* localized. If m is the mass of a particle whose position-vector is ρ, and velocity $\dot{\rho}$, the moment of its momentum in regard to the origin is $mV\rho\dot{\rho}$.



Since momenta are to be compounded like spins, it follows that the moment of the resultant of any number of momenta in regard to any point is equal to the sum of the moments of the components in regard to the same point.

We know that two spins about parallel axes, of the same magnitude but of opposite senses, compound into a

translation-velocity perpendicular to the plane containing them. This translation-velocity is equal to the sum of their moments in regard to any point in space, and its magnitude is the product of the magnitude of either spin into the distance between their axes. Two such spins are therefore equivalent to any other two equal and opposite spins about parallel axes in any plane parallel to that of the first pair, provided that this product is the same for the two pairs.

In the same way, if the momenta of two particles are on two parallel lines, equal in magnitude, but opposite in direction, the sum of their moments is the same in regard to every point of space, and the two momenta are equivalent to any other such pair the sum of whose moments in regard to all points is the same as that of the first pair. They must therefore be regarded as together constituting a *vector* quantity, of the nature of a moment of momentum; namely, it is precisely this constant sum of their moments.

If we start with two opposite momenta along parallel lines, which are only *nearly* equal in magnitude, the effect of making them more nearly equal is to send their resultant farther off and to diminish its magnitude. Hence a moment of momentum may be regarded as the limit of a very small momentum along a line which is very far off; just as a translation-velocity may be regarded as an infinitely small spin about an infinitely distant axis.

Every momentum may be resolved into an equal and parallel momentum through the origin, together with its moment in regard to the origin. Thus if the particle m at the end of the vector ρ have the velocity $\dot{\rho}$, its momentum is equivalent to $m\dot{\rho}$ through the origin together with the moment of momentum $mV\rho\dot{\rho}$.

It is sometimes convenient to indicate a rotor through the origin by prefixing the letter Ω to the symbol of a vector having the same magnitude and direction. Thus if α is any vector, $\Omega\alpha$ means the rotor through the origin which is one way of representing that vector. In this notation, the momentum of the particle whose mass-vector is $m\rho$ will be denoted by $\Omega m\dot{\rho} + mV\rho\dot{\rho}$. And similarly the velocity of a rigid body which has a spin θ about an axis

passing through the end of the vector ρ will be denoted by $\Omega\theta + V\dot{\rho}\theta$. In general, since any system of rotors and vectors is equivalent to a rotor through the origin and a vector, the expression for the resultant *motor* of such a system is $\Omega\alpha + \beta$.

The moment of a vector in regard to any point is simply the vector itself. Hence the moment of the system $\Omega\alpha + \beta$, in regard to the point whose position-vector is γ, is $V\alpha\gamma + \beta$.

ROTOR PART OF MOMENTUM.

When we resolve the momentum of every particle of a body into a parallel momentum through the origin, together with a moment of momentum, and then add the results for the whole body; the rotor part of this is independent of the particular origin we take. For it is simply the sum of all the momenta regarded as vectors.

We shall prove that *this rotor part is equal* (in magnitude and direction) *to the momentum of the resultant mass;* that is to say, it is the same as if the whole mass were collected at the mass-centre, and moving with its velocity. To prove this, it is only necessary to remember that the mass-vector of the resultant mass is the sum of the mass-vectors of all the particles, or it is $\int \rho dm$. The momentum of the resultant mass is the rate of change of this, or $\int \dot{\rho} dm$; but this is the vector-sum of the momenta of all the particles.

Now every motion of a body can be resolved into two; a translation equal to the velocity of the mass-centre, and the motion relative to the mass-centre. The momentum of the whole body is compounded of the momenta due to these motions. The momentum of the translation is a pure rotor, passing through the mass-centre; the proposition just proved shews us that *the momentum of the motion relative to the mass-centre is a pure vector.* For the rotor part of the whole momentum has been shewn to be equal to the momentum of the translation.

In calculating the momentum of any motion, therefore, it is sufficient to find the moment of momentum about the

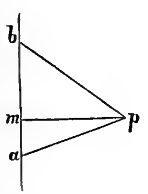
mass-centre; for since the momentum of motion relative to the mass-centre is a pure vector, its moment is the same in regard to every point of space.

MOMENTUM OF SPINS ABOUT FIXED POINT.

We now consider a rigid body rotating about a fixed point, and propose to calculate its moment of momentum in regard to the point.

Every spin about an axis through the point may be resolved into three component spins about the principal axes; and its moment of momentum will be the vector-sum of the moments of momenta due to these. The first thing to be done, therefore, is to find the moment of momentum of a spin about a principal axis.

In the first place we may remark that the *component along the axis* of the moment of momentum of a spin about *any* straight line, is equal to the angular velocity multiplied by the second moment of the body about that line.



For let a be the origin, ab the spin, p a rotating particle. The moment of its momentum about a is $i\,.\,ap$ multiplied by the magnitude of the momentum. Hence the component of this along ab is $i\,.\,mp$ multiplied by the same magnitude, which is $ab\,.\,mp$ into the mass of the particle. Thus the component is $ab\,.\,mp^2$ into the mass of the particle, or the angular velocity multiplied by the second moment. Adding this result for all the particles, we have the proposition as stated for the whole body.

The component in any direction perpendicular to the axis is the angular velocity multiplied by the mixed moment of two planes, one of which is the plane through a perpendicular to the axis, and the other is the plane through the axis perpendicular to the given direction. For the whole component perpendicular to the axis is $i\,.\,am \times$ magnitude of momentum; that is, it is parallel to pm, and of magnitude $ab\,.\,am\,.\,pm$. Now am is the distance of p from the normal plane through a, and the

component of pm in any direction perpendicular to ab is the distance of p from a plane through ab perpendicular to that direction; whence the proposition follows.

Now if the axis is a principal axis, the mixed moment of a plane through it and a plane perpendicular to it is zero. Hence *the moment about any point of momentum due to a spin about a principal axis through the point is directed along the axis, and is equal in magnitude to the angular velocity multiplied by the second moment.*

We may put these propositions together as follows. Let oX, oY, oZ be three rectangular lines through o, and let us consider a unit spin about oZ, which is denoted by k. The velocity of the end of $\rho, = xi + yj + zk$, due to this spin, is $Vk(xi + yj + zk), = xj - yi$. Let δm be the mass of the particle there situated, then the moment of its momentum is

$$V(xi + yj + zk)(xj - yi)\,\delta m = \{-xz\,.\,i - yz\,.\,j + (x^2 + y^2)\,k\}\,\delta m.$$

Hence the moment of momentum of the whole body is

$$-\int xzdm\,.\,i - \int yzdm\,.\,j + \int(x^2 + y^2)\,dm\,.\,k,$$

which gives the components as above stated.

Now let the axes be principal axes, and let the swing-radii about them be a, b, c. Then if the body have a spin $\theta, = pi + qj + rk$, its moment of momentum about the origin will be $m(a^2pi + b^2qj + c^2rk)$; since it is the resultant of the moments of momentum due to the three spins pi, qj, rk. The components of the spin must therefore be respectively multiplied by ma^2, mb^2, mc^2, to give the components of the moment of momentum.

Thus *the moment of momentum is a pure linear function of the spin.* The ellipsoid representing this function must have its semi-axes inversely proportional to the square roots of the second moments ma^2, mb^2, mc^2, that is, inversely proportional to a, b, c. Their actual values, in accordance with the formula [p. 176], are mbc, mca, mab each divided by π. Thus the relation between x, y, z for any point on this ellipsoid is

$$\pi^2(a^2x^2 + b^2y^2 + c^2z^2) = (mabc)^2.$$

The ellipsoid thus determined is called the *momental ellipsoid.*

There is no great convenience in thus fixing the *size* of the momental ellipsoid, on account of the two scales of representation involved; angular velocity is represented by a line, and mass by the inverse of a line (since mbc is one semi-axis). If we take any ellipsoid *similar* to this one, we may so choose the scale on which angular velocities are represented that the momentum due to a spin represented by any radius vector of the ellipsoid shall be equal to the mass of the body multiplied by the area of the conjugate section. We shall therefore give the name "momental ellipsoid" to any ellipsoid having axes along oX, oY, oZ inversely proportional to a, b, c; so that we may determine its size in particular cases by other considerations of convenience.

The momental ellipsoid is the reciprocal of the ellipsoid of gyration. For the ellipsoid of gyration has semi-axes a, b, c, and therefore if p be a perpendicular from the origin on a tangent plane to it, and make angles $\alpha\beta\gamma$ with the axes, we must have $p^2 = a^2 \cos^2\alpha + b^2 \cos^2\beta + c^2 \cos^2\gamma$. Now let us measure on the line p a distance r from the origin, such that $pr = C$, and let s be the point at the end of this distance, so that $os = r$. Then the locus of s will be the reciprocal of the ellipsoid of gyration [p. 101]. The coordinates of s are

$$x = r \cos\alpha, \quad y = r\cos\beta, \quad z = r\cos\gamma,$$

or
$$px = C\cos\alpha,\ py = C\cos\beta,\ pz = C\cos\gamma.$$

Therefore

$$p^2(a^2x^2 + b^2y^2 + c^2z^2) = C^2(a^2\cos^2\alpha + b^2\cos^2\beta + c^2\cos^2\gamma) = C^2p^2$$

and consequently $\quad a^2x^2 + b^2y^2 + c^2z^2 = C^2$

so that the locus of s is an ellipsoid whose axes are inversely proportional to a, b, c, that is, an ellipsoid of momentum.

MOMENTUM OF TWIST.

Let us now take the mass-centre of a rigid body for origin. The instantaneous state of motion of the body is made up of a translation-velocity equal to the velocity of the mass-centre, and a spin about some axis through it. Let these, regarded as vectors, be denoted by α and β respectively; then to indicate that the latter is really a rotor passing through the origin, we may denote the whole motion of the body by $\Omega\beta + \alpha$. This is the general symbol for a twist-velocity.

It is important to observe that the vector part of this velocity gives rise to the rotor part of the momentum, and vice versa. The translation is not localized, but its momentum has to pass through the mass-centre. The rotation is about a definite axis; but its momentum is a mere moment of momentum, which has only magnitude and direction, no definite position.

Thus the momentum due to the translation α is $\Omega m\alpha$, where m is the mass of the body. The momentum due to the spin $\Omega\beta$ is $\phi\beta$, where ϕ is the pure function which converts any spin into its moment of momentum. Hence the whole momentum due to the twist $\Omega\beta + \alpha$ is

$$\Omega m\alpha + \phi\beta.$$

Given any screw, the momentum due to a twist about it may be associated with another screw. Namely the momentum is equivalent to a rotor part along a certain axis, together with a vector part parallel to the axis; the proof of this is precisely like that for twists [p. 126].

We now propose to determine in what cases these two screws are identical. When this is so, the momentum $\Omega m\alpha + \phi\beta$ is a numerical multiple of the velocity $\Omega\beta + \alpha$;

say $$\Omega m\alpha + \phi\beta = x\,(\Omega\beta + \alpha),$$

whence $$m\alpha = x\beta,\ \ \phi\beta = x\alpha,$$

and therefore $$m\phi\beta = x^2\,\beta.$$

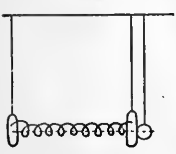
Hence the spin β must be about one of the principal axes at the mass-centre. Suppose it to be oX, then $\phi\beta = ma^2\beta$, and $x = \pm\, ma$. Consequently $\alpha = \pm\, a\beta$, and the pitch is $\pm\, a$. There are therefore six screws having this property; they are called the *principal screws* of the body. These axes are the principal axes at the mass-centre, and their pitches are the swing-radii about them.

APPENDIX I. (A.)

ACCELERATION DEPENDING ON STRAIN.

Imagine a spiral spring, like that of a spring-balance, with circular disks fastened to its ends and a small hole through one of them. Let it be partially compressed, and held in that position by a string attaching one of the two disks, through the hole in the other, to a bullet on the other side.

Hang up this spring in a horizontal position, by thin strings fastened to the disks. Let the bullet also be hung by a thin vertical string. Now cut the string which fastens the bullet to the further disk. The spring will then open and the bullet will move away. It will begin to move with a certain acceleration, depending upon the compression of the spring.



Now suppose the spring to be hung up to the same support as the bullet, so that the two may swing in contact, in a direction perpendicular to the axis of the spring; the spring being compressed as before. At any instant of the motion, let the string be again cut. Then the spring will begin to open, and the bullet to move away; but it will always begin to move with the *same* acceleration in the direction of the spring, provided that the compression is always the same.

If we allow the spring and bullet to swing in the direction of the spring's axis, the bullet will have an acceleration in the direction of the axis except when it is passing through the lowest point of its swing. If the string be cut at that instant, the acceleration of the bullet will be the same as before. But if the string be cut at any other instant, the difference between the acceleration before and after that instant will be precisely the acceleration with which the bullet began to move away from rest.

The greater the compression of the spring, the greater will be this acceleration. The method of finding the acceleration when the compression is given will be subsequently investigated.

We learn from these experiments that under certain circumstances a body A (the bullet) in contact with a strained body B (the spring) has an acceleration depending upon the strain of B, but wholly independent of the velocity of A or B. We have supposed the bullet and string to be moving with the same velocity, in order to make sure that the strain of the spring was always the same. If however the same condition of strain could be secured in any other way, the acceleration would be found quite independent of the velocity of the spring.

MASS.

If we now cut the bullet in two, and use one half in the same way, we shall find that for any given state of strain of the spring the acceleration is double what it was before. And generally, if we use any other piece of lead we shall find that the accelerations in the two cases are inversely proportional to the volumes of the pieces of lead; or, we may say, to the quantities of lead.

The same thing is true if we take different pieces of wood, or of any other substance, provided that the substance is *homogeneous*, that is, of the same nature all through. But if we compare the accelerations of a piece of wood and a piece of lead, we shall find that they are by no means inversely proportional to the volumes of the two bodies. A piece of wood, whose acceleration in the same circumstances is the same as that of a piece of lead, will be very much larger than the lead.

Let us now take an arbitrary body, say a certain piece of platinum. Of every other sort of substance, as of lead, wood, iron, etc., let us find a piece which under the same circumstances (i.e. with the same strain of the spring) has the same acceleration as the piece of platinum, and let each of these be called the *unit of quantity* of the substance

of which it is composed. Then if we consider a homogeneous body composed of one of these substances, the number of units of quantity which it contains is called the *mass* or measure of the body*. It is evidently the same as the number of units of quantity of platinum in a lump which has the same acceleration as the given body.

The piece of platinum actually† used is called the "kilogramme des archives," and is preserved in Paris. For convenience the unit of quantity is taken to be one-thousandth part of this, and is called a *gram*. Thus we may define:—

The mass of a body is the number of grams of platinum in a lump which under the same circumstances has the same acceleration as the given body.

If the body is not homogeneous, but is made up of parts of different kinds, the mass of the body is the sum of the masses of the parts.

The mass of a cubic centimetre of any substance is called the *density* of that substance.

We have supposed the densities of different substances to be measured by means of a certain spiral spring. We should, however, find exactly the same densities if we had used any other strained body.

The product of the mass of a body by its acceleration shall be called for shortness the mass-acceleration. Since we have found that the accelerations of any two bodies, in a given state of strain of the spring, are inversely proportional to their masses, it follows that the mass-acceleration of all bodies in contact with the spring in a given state of strain, is the same. This mass-acceleration is called ‡ the *stress* belonging to that state of strain.

In all this we have supposed the motion of the body to be pure *translation*.

The product of the mass of a body by its velocity is

* [contrast with definition on p. 1.]

† [? theoretically.]

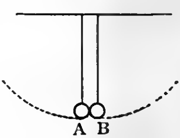
‡ [rather 'taken as the measure of'.]

called its *momentum.* Thus the mass-acceleration is the rate of change of momentum. Both momentum and mass-acceleration are *directed* quantities.

LAW OF COMBINATION.

If a body be in contact at the same moment with two strained bodies, its acceleration is the resultant of the accelerations which it would have when placed in contact with the two bodies separately. And in general, if a body be in contact with any number of strained bodies, its actual acceleration is the resultant of the accelerations due to the several bodies. Here again we assume that the whole motion is one of pure translation, or that the body may be regarded as a *particle.*

LAW OF RECIPROCITY.



Suppose two ivory balls, A, B, to be hung up side by side; let A be pulled away and then let go so as to impinge on B. The velocity of A will appear to be suddenly changed, and B will appear to suddenly acquire a velocity. The change, however, is not really sudden. At the moment of contact A has a certain velocity, but no tangential acceleration; B has neither velocity nor acceleration. After the contact, both bullets become compressed in the neighbourhood of contact; and then A has at every instant an acceleration opposite to its velocity, depending on the strain of B, while B has an acceleration in the direction of A's velocity, depending on the strain of A. After the compression has attained a certain magnitude, the compressed parts begin to expand again; and so long as any strain remains, each of them has an acceleration depending on the strain of the other. The two strains cease at the same moment, and then the bullets separate. But the whole time during which they are in contact is too short to be perceived by ordinary means.

During all this time, however, *the mass-acceleration of* A, *due to the strain of* B, *is equal and opposite to the mass-acceleration of* B, *due to the strain of* A. And the result is that the change of momentum of A during the contact is equal and opposite to the change of momentum of B. Or, as we may otherwise state it, the sum of the momenta of the two balls is the same before and after the impact.

The same thing holds good in the case of the bullet and spiral spring which we previously considered. In that case, however, the different parts of the spring are moving with different velocities and accelerations. But if we reckon the mass-acceleration of the whole spring as the sum of the mass-accelerations of its parts, it will still be true that at every instant the mass-acceleration of the bullet, due to strain of the spring, is equal and opposite to the mass-acceleration of the spring, due to strain of the bullet. For there is a slight compression, both of the bullet and of the disk with which it is in contact.

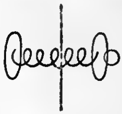
If we shorten the spring, by cutting off a part from the end away from the bullet, we shall make no difference to the acceleration of the bullet, provided that the remaining part is kept in the same state of strain as before. The only difference will be that this acceleration will diminish more rapidly and last a much shorter time; so that the bullet will acquire on the whole a less velocity. And finally if we remove the spring altogether, leaving only the disk at the end; and suppose the string stretched in any other way so as to produce the same compression of the disk as before*, and then to be suddenly cut; the acceleration of the bullet will be the same. But in this case it will exist only for an imperceptible time, during which the bullet will acquire only a very small velocity.

In general, the mass-acceleration due to the strain of two bodies in contact depends only on the strain of each *at the surface* of contact. No body can have a strain at its surface unless it is in contact with another body.

If we draw an ideal surface separating a body into two parts, each of these parts has a mass-acceleration due to

* [Is the disk compressed?]

the strain at the surface of separation. For example, if we divide the spiral screw by an imaginary plane of section, the mass-acceleration of the portion to the left of the plane is equal and opposite to that of the portion to the right together with the bullet, and each is due to the strain of the spring at the point of section. This mass-acceleration is called the *stress across the section.*



When it is *away* from the section on both sides, it is called *pressure;* when it is towards the section, it is called *tension.* Thus we see that what takes place at the common surface of two bodies in contact is a particular case of what takes place throughout the interior of any body.

GRAVITY.

When bodies are let go in the open air, they fall with more or less rapidity to the ground. This difference of velocity is found to depend on the presence of the air; and in the exhausted receiver of an air-pump the most different bodies fall through the same distance in the same time; having, as we remarked before, a constant acceleration of 981 centimetres a second per second. Thus every body left free *in vacuo* has a mass-acceleration vertically downwards, proportional to its mass.

The acceleration of the Moon is found to be very approximately the resultant of two accelerations, one directed towards the Earth, and the other towards the Sun. The acceleration towards the Earth is about one 3600th part of the acceleration of gravity, but varies within certain limits, being inversely as the square of the distance from the Earth's centre. Now the Moon is distant from that centre on the average about 60 times the Earth's radius, which is the distance from the centre of bodies on the Earth's surface. Hence the acceleration of the Moon towards the Earth is the same as that of any terrestrial body would be at the distance of the Moon, supposing the acceleration of the terrestrial body to vary inversely as the distance from the Earth's centre. Thus the Moon is to be regarded as a *falling body.*

Its acceleration towards the Sun is inversely as the square of the distance from the Sun; but although that distance is 400 times the distance from the Earth, this acceleration is always greater than the other, so that the orbit of the Moon is everywhere concave to the Sun.

The acceleration of the Earth is very approximately the resultant of two accelerations, one towards the Moon and one towards the Sun, both inversely as the square of the distance. The accelerations of the Earth and the Moon towards the Sun are equal at the same distance.

These descriptions of the acceleration of the Earth and Moon are only approximate, because each of them has other components, directed towards the planets, and inversely as the squares of the distances from them.

By the experiments of Cavendish it is shewn that bodies on the Earth's surface have accelerations towards each other which vary inversely as the squares of their distances; but these accelerations are very small and difficult to observe. The accelerations of all bodies towards a body A are equal at the same distance, but the accelerations at the same distance towards two bodies A and B are directly proportional to their masses.

Since then the acceleration of B towards A is to the acceleration of A towards B as the mass of A to the mass of B, it follows that the mass-acceleration of A is equal and opposite to the mass-acceleration of B, and each of them bears a fixed ratio to the product of the masses divided by the square of the distance.

In this case the mass-acceleration of a body depends, not upon the *strain* of an *adjacent* body as before, but upon the *position* of a *distant* body. The mass-acceleration in this case is called *attraction*, namely, the attraction of gravity. And we may now state the proposition enunciated by Newton, that *every particle of matter attracts every other particle, with an attraction proportional to the product of their masses divided by the square of the distance.*

Newton assumed that the Law of Reciprocity was true in the case of attraction, because he had proved it true by

experiment in the case of the pressure of adjacent bodies. When the Law of Reciprocity is assumed, it is sufficient to shew that the mass-acceleration of all bodies, due to any one (say the Earth), is the same at the same distance. This Newton did by his experiments on pendulums made of different substances, and by comparing the acceleration of the Moon with that of a falling body.

(B.)

ELECTRICITY.

If we rub a rod of glass with a piece of silk, and then touch with the glass rod two pith balls hung near one another by silk threads, they will move away from one another. The same thing happens if we touch both of them with the silk. But if we touch one with the silk and the other with the rod, they will move together until they come into contact, after which they will hang down as before.

This is commonly described by saying that both glass and silk acquire a certain *charge* of electricity, one positive and the other negative, which is partially communicated to the pith balls. Two bodies having *like* charges (both positive or both negative) move away from one another when free; two bodies having *unlike* charges (one positive and the other negative) move towards one another.

In either case the mass-accelerations of the two bodies are equal and opposite, and each is proportional to the product of the charges divided by the square of the distance. This mass-acceleration is called *attraction* or *repulsion*, accordingly as the bodies approach, or recede from, one another.

When a charge is communicated to a piece of metal supported on a glass rod, it distributes itself all over the surface of the metal, so that it must have gone from one part of it to another. A body which admits of this travel-

ling is called a *conductor*. Other bodies [which do not admit of this travelling], such as glass, are called *insulators*.

When two conductors are placed near one another, and one or both of them charged, there is found a certain *distribution* of charge over the surface of both, which is the same as it would be if each element of charge had an acceleration compounded of accelerations *from* all other elements of the same kind and *to* all elements of different kinds, proportional to their magnitudes divided by the square of the distance, while those elements which are on the surface of the conductor have also a normal acceleration inwards, equal and opposite to the normal component of the resultant of all the other accelerations. Thus we see that any two elements of charge have *charge-accelerations* which are equal and opposite, and proportional to the product of the charges divided by the square of the distance.

We do not call these *charge-accelerations* attraction or repulsion, because there is reason to think that a charge is not a body, but a strain or displacement of something which freely pervades the interstices of bodies.

MAGNETISM.

A kind of iron ore, called *loadstone*, is found to attract pieces of iron placed near it. This property may be communicated to iron by rubbing it with loadstone; the piece of iron is then said to be *magnetized* and is called a *magnet*. If a long thin bar be uniformly magnetized and hung by its centre, it will point nearly north and south; the end towards the north is called the *north pole* of the magnet, the other end the *south pole*. Two such bars placed in the same straight line, north and south, in their natural positions, will have accelerations towards each other inversely proportional to their masses; the mass-accelerations are equal and opposite, and each is the resultant of four. Namely the mass-acceleration of SN is compounded of mass-accelerations *towards* $S'N'$, inversely as the squares of the distances NS', SN': and of mass-accelerations *away from*

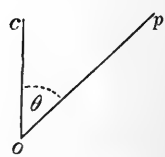
$S'N'$, inversely as the squares of the distances SS', NN'. Thus we may say that there is *attraction* between a north and a south pole, and *repulsion* between two north or two south poles, proportional in each case to the product of their strengths divided by the square of the distance. The two poles of any one magnet are always of the same strength.

In any other position of the two magnets, each of them has an angular acceleration, or tends to turn round.

ELECTRIC CURRENTS.

A copper wire connecting two plates, of copper and zinc respectively, placed in dilute sulphuric acid, is found to be in a peculiar state, which is described by saying that it *carries an electric current.* If the wire be divided, the two ends are found to be oppositely electrified; and when they are brought together again, there is a continuous passage of electric charge from one to the other. There are other substances besides copper which will carry an electric current, and other modes of producing it besides the arrangement just described, which is called a *battery.*

When a small magnet hung by its centre is brought near a wire carrying a current, it places itself at right angles to the direction of the wire. When we come to consider the motion of a rigid body having different accelerations in its different parts, we shall be able to shew that the flux of the velocity-system of the magnet is equivalent to two mass-accelerations of its two poles in different directions. These follow the following law. Let oc be a line in the direction of a small piece of wire at o carrying a current, and of length representing the product of the length of the wire by the strength of the current. Then the mass-acceleration of a magnetic north pole at p will be perpendicular to the plane poc, proportional to $oc \sin\theta : op^2$, or, which is the same thing, proportional to the vector product of oc and op divided by the cube of the distance op.



The mass-acceleration of each of two small pieces of wire carrying currents due to the position of the other is not certainly known, as it is only possible to experiment upon closed circuits. The law of dependence on the position is too complicated in this case to be explained at present; but it agrees with the other cases which we have examined in these important respects:—The mass-acceleration of each conductor depends on the *position* of the other and the strength of the two currents, and the two mass-accelerations are equal and opposite.

LAW OF FORCE.

We have now briefly examined various cases of mass-acceleration or rate of change in the momentum of a body. We have found that it depends upon one of two things: the strain of an adjacent body, or the position and state of a distant body. But it does not depend, in any case which we have examined, on the *velocity* either of the body itself or of other bodies.

This quantity, then, the rate of change in the momentum of a body, may be calculated in two ways. First, by observing the motion of the body; in this case the quantity is called mass-acceleration, or flux of momentum. Secondly, by observing the strain of adjacent bodies and the position and state of distant bodies; when so calculated, it is called *force*. Force is a name given to the flux of momentum of a body, which is intended always to remind us that it depends partly on the strain of adjacent bodies and partly on the position and state of distant bodies.

In all cases the actual flux of momentum is the resultant of those which are severally due to the strains of different adjacent bodies and the position and state of different distant bodies.

When, for example, a book rests on a table, it has a mass-acceleration downwards equal to its mass multiplied by 981 centimetres a second per second. This is due to the position of the Earth, and depends on the mass of the Earth and the distance of the book from its centre. The book has also an equal mass-acceleration vertically up-

wards, due to the strain of the table, which is slightly compressed under it. If the Earth, excepting this table and book, could be suddenly annihilated, the book would begin to move upwards from the table with an acceleration of 981 centimetres a second per second. But this acceleration would diminish so very rapidly and disappear in so minute a time (the strains being very small) that the book would acquire on the whole only a very small velocity.

An electrified magnet suspended by an elastic string at its middle, in the presence of electrified and magnetic bodies, will have at every instant a mass-acceleration compounded of those due to the position of the Earth, the position and state of the electrified and magnetic bodies, and the strain of the elastic string. The composition takes place according to the already known laws of composition or addition of vectors and of rotors passing through the same point. It is understood that for the present the *rotation* of the magnet is neglected in this computation.

There are certain cases of apparent exception to the Law of Force which shall be here briefly mentioned. A bullet travelling through the air has a mass-acceleration opposite to its velocity, which varies according to a complicated law so long as the velocity is below the velocity of sound, but afterwards is nearly proportional to the square of the velocity. In this case, however, the mass-acceleration depends *directly* on the strain of the air, which itself depends on the velocity of the body during a short time previous to the moment considered; so that *indirectly* the mass-acceleration depends on the velocity of the body a little before. This mass-acceleration, depending on the strain of a fluid, is called *resistance*, or *fluid friction.*

Two solids in contact experience equal and opposite mass-accelerations, called *friction,* parallel to the surface of contact, which are independent of the velocity when they are once moving, but different from their values when the solids are relatively at rest. This kind of friction, like that of fluids, is really due to a shearing strain of the

surfaces in contact, and the difference between friction in rest and in motion is to be accounted for by a change in the nature of the surfaces.

The reader must be very careful to distinguish between the technical meaning of the word *force*, explained in this section, and the various meanings which the word has in conversation or in literature. He must especially learn to dissociate the dynamical meaning from the idea of muscular exertion and the feelings accompanying it. When I press any object with my hand, a very complex event takes place. As a consequence of a certain molecular disturbance in my brain, nervous discharges go to the muscles of my arm and hand. The effect upon the muscles is to produce an internal strain, in virtue of which my hand receives a certain mass-acceleration. The part of it in immediate contact with the object, and the object itself, are slightly compressed. The object has then a mass-acceleration due to this strain of an adjacent body. The compression of my hand, and the continued strain of the muscles, are followed by nerve-discharges which travel back to my brain, there to result in a further disturbance. Besides these physical facts, there coexist with the brain-disturbances the mental facts of a sensation of effort to push the object, and a sensation of pressure on the hand and of resistance to the effort. Now in the case of a bullet in contact with a strained spring, there is nothing to correspond, either with the nervous and cerebral mechanism, or with the sensations of effort and resistance. The only fact common to the two events is that the flux of momentum of a body A stands in a definite numerical relation with the strained condition of an adjacent body B. In one case A is the bullet and B the spring. In the other case A is the object pressed and B the surface of my hand.

The scientific meaning of the word *force* relates only to this common fact. The various literary and conversational meanings imply a reference, direct or by metaphor, to the complex structure of an organism, and the mental facts which accompany it.

GENERAL STATEMENT OF THE LAWS OF MOTION.

We may now put together the inductions from experience which we have lately described.

Definition of Mass. *The mass of a body is the number of grams of platinum in a lump of platinum which under the same circumstances* (contact with the same strained body in the same state of strain) *has the same acceleration as the given body.*

Law of Force. *The product of the mass and acceleration of a body which does not rotate depends partly upon the strain of adjacent bodies and partly upon the position and state of distant bodies; but not directly upon the velocity of any of them.*

Considered as so depending, this product of mass and acceleration is called *Force.* Force is the mass-acceleration of a body expressed in terms of the strain and position of other bodies.

Law of Composition. *The actual mass-acceleration of a body which does not rotate is the resultant, or vector-sum, of the mass-accelerations which it might have separately in virtue of the strains of the several adjacent bodies and the positions of the several distant bodies.*

We may also state this as follows. *Forces in a particle, or in a body which does not rotate, are compounded according to the law of addition of vectors, or of rotors passing through the same point.*

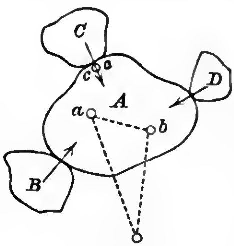
Law of Reciprocity. *The mass-acceleration of a body* A, *due to the strain or position of another body* B, *is equal and opposite to the mass-acceleration of* B *due to the strain or position of* A.

The mass-acceleration of A due to B will be called the force *in* A due to B. When we wish to mention the equal and opposite forces together, we shall speak of the forces *between* A and B.

D'ALEMBERT'S PRINCIPLE.

In all that precedes we have considered the change of momentum only of a body which either does not rotate, or is so small that its rotation may be neglected. We now go on to treat the case of rotating or straining bodies, by means of a theorem first explicitly stated by D'Alembert.

Consider a body A, in contact with various strained bodies B, C, D. Any small portion a of the body A will have a force (mass-acceleration) which is partly due to the strain of the adjacent parts of A, partly to the position of the distant parts, such as b, and partly to the position of distant bodies. If the small portion be in contact with a strained adjacent body, as at c, it will of course have a force due to that strain.



The forces in a which are due to the other parts of the body A, whether adjacent or distant, are called *internal forces.* Those which are due to other bodies, whether adjacent or distant, are called *external forces.*

Now if a and b are two parts so small that their rotation may be neglected, we know that the force in a due to b is equal and opposite to the force in b due to a; and this is true whether the two parts are adjacent or distant. Each of these forces then may be represented by the same length measured on the line ab, but measured in opposite senses.

The *moment* of a force (the mass-acceleration of a particle) about a point, is the mass of the particle multiplied by the moment of its acceleration. Suppose that the forces in a and b due to b and a respectively are represented by ab and ba; their moments about any point o will be the doubles of oab and oba, these areas being regarded as vectors. Hence the sum of the moments about any point o of the force in a due to b and the force in b due to a is zero.

We know also that the moment about o of the whole force in a is the vector-sum of the moments of the several forces of which it is the resultant. That is to say

$$\text{moment of whole force in } a = \text{moment of internal forces} + \text{moment of external forces}$$

and the same thing is true for every point of the body. Hence, adding all these equations together, we find

$$\left.\begin{array}{l}\text{sum of moments of forces}\\ \text{in all parts of } A\end{array}\right\} = \left\{\begin{array}{l}\text{sum of moments}\\ \text{of internal forces}\end{array}\right\} + \left\{\begin{array}{l}\text{sum of moments}\\ \text{of external forces.}\end{array}\right.$$

Now when we add together the moments of the internal forces of all the particles, they disappear in pairs, because the moment of the force in a due to b is equal and opposite to the moment of the force in b due to a. Hence the sum of the moments of all the internal forces is zero. Consequently we find that, about *any* point o whatever

$$\left.\begin{array}{l}\text{sum of moments of forces}\\ \text{in all parts of } A\end{array}\right\} = \left\{\begin{array}{l}\text{sum of moments}\\ \text{of external forces.}\end{array}\right.$$

It must be carefully observed that in the demonstration of this theory no assumption has been made about the nature of the body A; it may be rigid or elastic, solid, liquid, or gaseous. Only the rotation of very small parts of it has been neglected. This neglect will be subsequently remedied. It is implied in this that no part of the body has a force due to another part like that in a magnetic north pole due to an element of current, which is not in the line joining them; because the law of reciprocity cannot be understood in such a case without taking account of rotation.

(C.)

Now the doctrine that Force is at the bottom of things is essentially an attempt to answer the question "why?" For Force is defined to be any *cause* which changes or tends to change a body's state of rest or motion. To the question, "*why* does the body move in such a way?" would be answered "*because* it is acted on by such and such a force"—force being now* measured by the quantity of motion which it produces in unit of time.

We are not to conclude, however, that the doctrine of Force is a *mere* attempt to answer the question *why*. If the common definition included all the meaning of the word, the above question and answer might be expressed thus:—why does this body's motion change? because it is acted on by that which changes it—and so be either tautological or absolutely unmeaning, according to the doubt expressed above. But as a matter of fact the word has a connotation going much beyond this definition; it links together in itself two meanings, and this linking is at once the effect and the expression of belief in a great physical law. Before people had any clear ideas about force, if we had asked them what makes a thing move? they would have replied "other things". Aristotle, for instance, would no doubt have given this answer. After the step which substitutes for "other things" some virtue or property of these things—their attraction, repulsion, resistance—there still remains the idea that this property or virtue depends in some way upon the position, relative to the moving thing, of the bodies to which it belongs; that is to say, that there are some rules (whether we know them all or not) by which when the relative position of bodies is known, their *force* may be calculated; or, in mathematical phraseology, that force is a function of situation. The expression "force exercised by a body" comprehends as part of its meaning the notion of something that depends on the situation of the body relative to the thing moved by its force. When therefore it is held further that the

* Thomson and Tait, *Nat. Phil.* Art. 220.

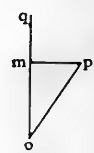
effect of force is not motion but the acceleration of motion, to which, therefore it is proportional; the binding up of these two ideas in the same word is equivalent to a statement of the following law, which for convenience may be called the law of *acceleration by place:*—

There are rules by which when the position of a body in relation to surrounding things is known, its acceleration may be calculated; and this acceleration is the same, with whatever velocity the body passes through that position.

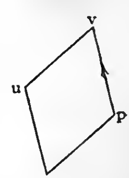
(D.)

THE ROTATION OF A RIGID BODY.

When a body has a fixed point o, the angular velocity of the body may be represented by a vector θ in the direction of the instantaneous axis, of length equal to the rotational velocity. It is so drawn that when we look back on it the rotation appears counter-clockwise. Thus a rotation represented by oq will cause p to move away from the eye.



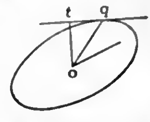
The resulting velocity of p *is the vector product of* oq, op, *or it is* $V\theta\rho$ *if* $\rho =$ op. For p moves in a circle of radius pm with angular velocity oq; therefore $\dot{p} = oq.pm$ perpendicular to the plane qop. But this is $V\theta\rho$. Or if $\theta = ip + jq + kr$, $\rho = ix + jy + kz$, then $\dot{\rho} = i(qz - ry) + j(rx - pz) + k(py - qx)$.



The moment about o *of the momentum of* p *is* $mV\rho\dot{\rho}$, or $mV\rho V\theta\rho$ since $\dot{\rho} = V\theta\rho$. For in general the moment of momentum of p is twice the area opv multiplied by the mass of p. But this is twice the area uop multiplied by m, or $mV\rho\dot{\rho}$ if $\rho = op$, $\dot{\rho} = pv = ou$. Expressing this in terms of the components, we find $V\rho\dot{\rho} = i(py^2 + pz^2 - rzx - qxy) + j(qz^2 + qx^2 - ryz - pxy) + k(rx^2 + ry^2 - pzx - qyz)$.

The moment of momentum of the whole body about o is

clearly $\int m V\rho\dot{\rho}$; or, putting $a = \int m\,(y^2 + z^2)$, $b = \int m\,(z^2 + x^2)$, $c = \int m\,(x^2 + y^2)$, $f = \int myz$, $g = \int mzx$, $h = \int mxy$ it is $\mu = i\,(ap - hq - gr) + j\,(-hp + bq - fr) + k\,(-gp - fq + cr) = \phi\,(\theta)$ say. Hence there is an ellipsoid with centre o such that if $\theta\,(= oq)$ is a vector to its surface, qt the tangent plane at q, ot the perpendicular upon it, then μ is parallel to ot and of magnitude ot^{-1}. This is called the *momental ellipsoid.* Its axes are a set of lines at right-angles through o such that $o = \int myz = \int mzx = \int mxy$; that is, they are the same as the axes of the swing-ellipsoid, or *principal axes* at o. If ijk coincide with these at the moment in question, then $\theta = ip + jq + kr$, $\mu = iap + jbq + kcr$.



By D'Alembert's principle the flux of the moment of momentum is equal to the moment of the external forces. Let this be ϖ, then $\dot{\mu} = \varpi$. The problem is, from this equation to find θ, and thence the position of the body, in terms of the time. But as $\dot{\mu}$ involves not only the flux of θ but also those of $abcfgh$, the equation in this form is unmanageable.

If however no forces act on the body, $\dot{\mu} = \varpi = o$, or μ is constant. Hence ot is fixed in magnitude and direction, and therefore the tangent plane qt is fixed in space. Since oq is the instantaneous axis of rotation, q has no velocity; therefore *the momental ellipsoid rolls on a fixed plane.* This representation of the motion is due to Poinsot.

MOVING AXES.

Let α, β, γ be three unit-vectors along the principal axes of the body, and consider the flux of any vector $\rho = \alpha x + \beta y + \gamma z$. This vector changes not only on account of the change in its components x, y, z, but also on account of the change in the unit-vectors α, β, γ. Let us first suppose that ρ moves with the body, so that x, y, z are constant; and that the body has an angular velocity $\theta = \alpha p + \beta q + \gamma r$. Then $\dot{\rho} = V\theta\rho$, or

$$\dot{\alpha} x + \dot{\beta} y + \dot{\gamma} z = \alpha\,(qz - ry) + \beta\,(rx - pz) + \gamma\,(py - qx);$$

whence $\dot{\alpha} = \beta r - \gamma q$, etc. Next let ρ move relatively to the

body, and let $(\dot{\rho})$ be the relative velocity, or $(\dot{\rho})=\alpha\dot{x}+\beta\dot{y}+\gamma\dot{z}$. Then the whole flux of $\dot{\rho}$ is the resultant of these two motions, or $\dot{\rho}=(\dot{\rho})+V\theta\rho$.

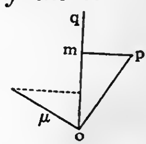
Applying this to the flux of the moment of momentum, we find $\varpi=\dot{\mu}=(\dot{\mu})+V\theta\mu$. Or, since the axes are principal axes and therefore $\mu=\alpha ap+\beta bq+\gamma cr$, where a, b, c are constant,

$$\alpha l+\beta m+\gamma n = \alpha\dot{p}+\beta\dot{q}+\gamma\dot{r}+\alpha(c-b)\,qr+\beta(a-c)\,rp+\gamma(b-a)\,pq.$$

Here l, m, n are the components of ϖ. The three equations here involved are called Euler's equations.

KINETIC ENERGY.

The kinetic energy of a particle is — half the scalar product of rotation and moment of momentum. For moment of momentum $=\mu$, perpendicular to op, of length $=m\,.\,op\,.\,pm\,.\,oq$. Hence —its scalar product with oq is $m\,.\,pm^2\,.\,oq^2$ or mv^2, which is twice the kinetic energy.



Therefore, the kinetic energy T of the whole body is

$$-\tfrac{1}{2}\,S\theta\mu=\tfrac{1}{2}\,(ap^2+bq^2+cr^2-2fqr-2grp-2hpq)$$

which when the axes are principal axes reduces to

$$\tfrac{1}{2}\,(ap^2+bq^2+cr^2).$$

If no forces act on the body T is a constant; this is equivalent to saying that the extremity of θ is always on a fixed ellipsoid with centre o. $[S\theta\mu+2T=0]$.

For solution of Euler's equation when $\varpi=o$, see Kirchhoff, 1 . 66. We have $\phi=am\,(nt+\epsilon)$, $p=P\cos\phi$, $q=Q\sin\phi$, $r=R\Delta\phi$; where

$$P^2=\frac{M^2-2cT}{a\,.\,a-c},\quad Q^2=\frac{M^2-2cT}{b\,.\,b-c},\quad R^2=\frac{2aT-M^2}{c\,.\,a-c},$$

$$n^2=\frac{\overline{2aT-M^2}\,.\,\overline{b-c}}{abc},\quad k^2=\frac{M^2-2cT}{2aT-M^2}\cdot\frac{a-b}{b-c},$$

where T is the kinetic energy and M the magnitude of moment of momentum.

TOP.

Let the axis of the top make an angle t with the vertical, and let the vertical plane through it make an angle u with a fixed vertical plane. Let γ be the vertical through the tip, α the projection of the axis; then the rotation of these axes $= \theta = \gamma\dot{u}$. Rotation of top

$$= \alpha \sin t \,.\, p + \beta\dot{t} + \gamma\,(p\cos t + \dot{u}).$$

Momentum $= \alpha cp \sin t + \beta a\dot{t} + \gamma\,(cp\cos t - a\dot{u}).$

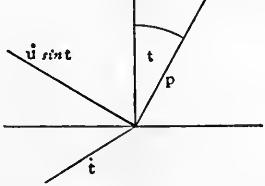
Also

$$\dot{\alpha} = \beta\dot{u},\ \ \dot{\beta} = -\alpha\dot{u},\ \ \dot{\gamma} = 0,\ \ \varpi = \beta gl\sin t = \alpha\,(cp\dot{t}\cos t - a\dot{t}\dot{u})$$
$$+ \beta\,(a\ddot{t} + cp\dot{u}\sin t) + \gamma\,(a\ddot{u} - cp\dot{t}\sin t).$$

Whence $cp\cos t = a\dot{u}$, and

$$a\ddot{t} = (-cp\dot{u} + gl)\sin t = \left(gl - \frac{c^2p^2}{a}\cos t\right)\sin t.$$

Momentum $= cp\,(\gamma\cos t + \alpha\sin t).$



Rotation of top

$$= p\,(\gamma\cos t + \alpha\sin t) + \dot{u}\sin t\,(\gamma\sin t - \alpha\cos t) + \beta\dot{t}.$$

Momentum

$$= cp\,(\gamma\cos t + \alpha\sin t) + a\dot{u}\sin t\,(\gamma\sin t - \alpha\cos t)$$
$$+ a\beta\dot{t},$$

$$\mu = \alpha\,(cp\sin t - a\dot{u}\sin t\cos t) + \beta a\dot{t} + E\gamma,$$

where

$$E = cp\cos t + a\dot{u}\sin^2 t \text{ which is constant.}$$

Now $\varpi = \dot{\mu}$ or,

$$\beta gh\sin t = \alpha\,(cp\dot{t}\cos t - a\ddot{u}\sin t\cos t - a\dot{u}\dot{t}\cos 2t - a\dot{u}\dot{t})$$
$$+ \beta\,(a\ddot{t} + cp\dot{u}\sin t - a\dot{u}^2\sin t\cos t)$$

$$+ 2T = [\text{terms erased.}]$$

$$H - 2gh\cos t = cp^2 + a\,(\dot{u}^2\sin^2 t + \dot{t}^2)$$

[side work.] $\qquad c\,(p + u\cos t)\,\alpha + au\sin t\left(\alpha - \dfrac{\gamma}{\cos t}\right)$

$$\dot{u}^2\sin^2 t = \frac{(E - cp\cos t)^2}{a^2\sin^2 t},$$

$$\therefore\; a\dot{t}^2 + \frac{(E - cp\cos t)^2}{a\sin^2 t} + 2gh\cos t + cp^2 - H = 0,$$

or if $\qquad \cos t = x,\; \dot{t}\sin t = -\dot{x}$

$$a\dot{x}^2 + (E - cpx)^2 + (2ghx + cp^2 - H)\,a\,(1 - x^2) = 0,$$

$$^*a\dot{x}^2 = 2ghax^3 + \{(cp^2 - H)\,a - c^2p^2\}\,x^2 + 2\,(Ecp - gha)\,x$$
$$- E^2 - cp^2a + Ha.$$

From $T + T'$, p. 445,

α = inclination of spiral,

a = radius of cylinder,

t = twist of wire,

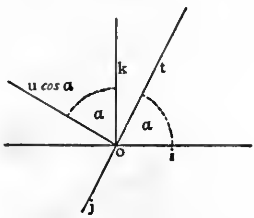
$\dfrac{\cos^2\alpha}{a}$ = curvature; $\therefore$ components of bending stress in plane perpendicular to axis θ in plane through axis are $B\dfrac{\cos^2\alpha}{a}\cos\alpha$ and $B\dfrac{\cos^2\alpha}{a}\sin\alpha$. Also components of twisting stress in same planes are $At\sin\alpha$ and $-At\cos\alpha$. Hence, for equilibrium

$$G = \frac{B\cos^2\alpha}{a}\cos\alpha + At\sin\alpha,$$

* [This line is in pencil; a second line also in pencil, is omitted.]

$$-Ra = \frac{B\cos^2\alpha}{a}\sin\alpha - At\cos\alpha.$$

Making $G=0$, we have $t = -\frac{1}{a}\frac{B}{A}\frac{\cos^3\alpha}{\sin\alpha}$, $R = \frac{At}{a\cos\alpha}$.



For rotation,

$$\theta = t\,(i\cos\alpha + k\sin\alpha) + u\cos\alpha\,(k\cos\alpha - i\sin\alpha)$$

$$\mu = At\,(i\cos\alpha + k\sin\alpha) + Bu\cos\alpha\,(k\cos\alpha - i\sin\alpha)$$

$$i = ju$$

$$jgh\cos\alpha = \varpi = \dot{\mu} = jAtu\cos\alpha - Bu^2\cos\alpha\sin\alpha,$$

or $gh = Atu - Bu^2\sin\alpha$, $\quad u = \frac{\cos\alpha}{a} - gh = R,$

$G =$ coefficient of k in $\mu = At\sin\alpha + Bu\cos^2\alpha.$

For steady motion

$$\dot{t} = o,\ \ddot{u} = o, \text{ and } gh\sin t = cp\dot{u}\sin t - a\dot{u}^2\sin t\cos t$$

or $$cp\dot{u} = gh + a\dot{u}^2\cos t.$$

Suppose now that the top has a rotation p about α the axis of it, and a rotation u about γ the vertical; $\beta = V\alpha\gamma$, and the whole rotation $p + u\cos t$ about the axis is constant; say $p + u\cos t = u$. Then whole rotation $= p\alpha + \dot{t}\beta + u\gamma$, and momentum

$$= cp\alpha + a\dot{t}\beta + (c\cos^2 t + a\sin^2 t)\,u\gamma.$$

$$u = \frac{\cos\alpha}{r},\ p = \tau, \text{ curvature } = u\cos\alpha = \frac{\cos^2\alpha}{r}.$$

Momentum about vertical

$$E = cp\cos t + au\sin^2 t = cp\sin\alpha + au\cos^2\alpha$$
$$= cp\sin\alpha + a\frac{\cos^3\alpha}{r}.$$

Flux of momentum about horizontal

$$R = cpu - au^2\cos t = cp\frac{\cos\alpha}{r} - a\frac{\cos^2\alpha}{r^2}\sin\alpha.$$

The two equations are $cpu = gh + au^2\cos t$,

and $$E = cp\cos t + au\sin^2 t.$$

(E.)

If $\sigma\sigma_1$ be two different velocities, imagined as belonging to a particle m, the scalar product of either into the momentum of the other is the same, namely $mS\sigma\sigma_1$.

When this quantity vanishes, the energy of the resultant motion is equal to the sum of the energies of the components, and the motions are called *conjugate.*

In general, the energy of the resultant is less than the sum of the energies of the components by twice the scalar product of either velocity into the momentum of the other.

Let now QR be two velocity systems, imagined as belonging to a body m, in the same situation. The momenta belonging to them may be called ϕQ, ϕR. If we calculate for each particle the scalar product of its velocity in either system by its momentum in the other, and add all the results, we shall obtain a quantity which may be called the scalar product of the velocity of one motion into the momentum of the other, and denoted by $S.Q\phi R$ or $S.R\phi Q$. It is clear from the case of the particle that these expressions are always identical.

When this quantity vanishes, the energy of the resultant of the two motions is equal to the sum of the energies of the components, and the motions are called *conjugate.*

In general, the energy of the resultant of any number of motions is equal to the sum of their energies less twice the scalar products of their momenta and velocities, two and two.

If the body have n degrees of freedom, its situation may be determined by n quantities $q, r, \ldots$ Let Q be that velocity which the body has when $\dot{q}=1$ and all the other variables are constant, R that which it has when $\dot{r}=1$ and the rest are constant, etc. Then $\partial_r Q = \partial_q R$. For if ρ be the position-vector of a particle, its velocity in the system Q is $\partial_q \rho$, and in the system R it is $\partial_r \rho$. The part of $\partial_r Q$ belonging to this particle is $\partial_q \partial_r \rho$, which is also the part of $\partial_q R$ belonging to it. Similarly for all particles.

The energy T of any motion U is given by the equation $-2T = \Sigma S.Q\phi R\dot{q}\dot{r}$, where for q, r are put independently all the variables. For the motion U is made up of $\dot{q}Q, \dot{r}R, \ldots$ and its energy is consequently calculated by the rule stated above.

Hence $-\delta_{\dot{q}}T = \Sigma S.Q\phi R.\dot{r} = S.Q\Sigma\phi R.\dot{r} = S.Q\phi U$ where ϕU is the whole momentum.

The motions Q and U depend on the variables q and t. Hence $\partial_q U = \dot{Q}$. Now since

$-2T = S.U\phi U, -2\partial_q T = S.\partial_q U\phi U + S.U\phi\partial_q U = 2S.\partial_q U\phi U$;

therefore $\partial_q T = S.\dot{Q}\phi U$.

(F.)

Momentum of particle = mass × velocity.

But mass being in a definite place, this is a *localized vector* or *rotor*.

Momenta of two particles to be compounded by rules for rotors. This is convention, to be justified by results.

Hence resultant momentum of body having translation velocity passes through mass-centre.

Def.: moment of momentum, moment of resultant = sum of moments of components.

Momentum of particle m at end of ρ, having velocity σ, is equivalent to momentum $m\sigma$ through origin + *moment of momentum* $mV\rho\sigma$. This is a *vector*, not localized at all. Hence any system of momenta is equivalent to a momentum along a certain axis together with moment of momentum in plane perpendicular to that axis.

To determine momentum due to rotation about axis ω through the origin. Velocity of end of ρ is $V\omega\rho$; hence momentum $= \int V\omega\rho dm$ through origin + moment of momentum $\int V.\rho V\omega\rho dm$. Now $\int V\omega\rho dm = V.\omega\int\rho dm$, which is the momentum of the resultant mass. Hence if axis passes through mass-centre, momentum is pure vector. Now we know that whole momentum = momentum of translation velocity equal to that of mass-centre + momentum of rotation about mass-centre, hence the former is the rotor part of the whole momentum.

Moment of momentum of rotation about axis through mass-centre is the same for every point, we therefore now consider moment of momentum in regard to mass-centre of rotation about principal axis oz. Velocity of $ix + jy$ is $-iy + jx$, moment $= V(ix + jy + kz)(-iy + jx)$. For whole body this is $= k\int(x^2 + y^2)dm - i\int xzdm - j\int yzdm$, $= Ak$, and therefore parallel to axis of rotation when that is a principal axis.

Hence for rotation θ, moment of momentum $= \phi(\theta)$, where $\phi = \begin{vmatrix} A00 \\ 0B0 \\ 00C \end{vmatrix}$. The ellipsoid of this function is called the *momental ellipsoid;* it is reciprocal to the ellipsoid of gyration.

If therefore mass-centre has velocity σ, and there is rotation ω about axis through it, the momentum is rotor $m\sigma$ together with vector $\phi\omega$.

Let $\phi\rho = exi + fyj + gzk$,

$$V\rho\phi\rho = \overline{f-g}.yzi + \overline{g-e}.zxj + \overline{e-f}.xyk.$$

Of course $\int\phi\rho dm = m\phi\bar{\rho}$.

Next, $\iint \frac{\mu}{r^2} . r^2 d\Omega dr = \int \mu r d\Omega$; thus, in a squirt, if we take a small cone, vertex at the source, the momentum is uniformly distributed along its axis.

Energy of translation $= \frac{1}{2} mv^2$.

Energy of rotation

$$V\omega\rho = (qz - ry)\, i + (rx - pz)\, j + (py - qx)\, k,$$

$$-\tfrac{1}{2}\int \sigma^2 dm = \tfrac{1}{2}\{p^2 \int (y^2 + z^2)\, dm + q^2 \int (z^2 + x^2)\, dm + r^2 \int (x^2 + y^2)\, dm\}$$
$$= \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2) = -\tfrac{1}{2} S\omega\phi\omega,$$

for two rotations ω, θ

$$-S . V\omega\rho V\theta\rho$$
$$= (qz - ry)(q_1 z - r_1 y) + (rx - pz)(r_1 x - p_1 z)$$
$$+ (py - qx)(p_1 y - q_1 x) + \int S . V\omega\rho V\theta\rho dm$$
$$= S\omega\phi\theta = S\theta\phi\omega,$$

for if $\qquad \int (V\omega\rho)^2 dm = S\omega\phi\omega,$

then $\qquad \int (V\omega\rho + V\theta\rho)^2 dm = S(\omega + \theta)\, \phi\, (\omega + \theta),$

whence $\quad 2\int S . V\omega\rho V\theta\rho dm = S\omega\phi\theta + S\theta\phi\omega = 2S\omega\phi\theta.$

Two motions in which the velocity of a particle δm is σ and σ_1 respectively are called *conjugate* when $\int S\sigma\sigma_1 dm = 0$, or when the scalar product of the velocity of one by the momentum of the other vanishes. In that case the energy of their resultant is equal to the sum of their energies.

Two rotations about axes which meet are conjugate when the axes are conjugate in regard to the momental ellipsoid at their point of intersection.

Consider motions $m\sigma_1$ and ω_1, $m\sigma_2$ and ω_2; their scalar product is $mS\sigma_1\sigma_2 + \omega_1\phi\omega_2$ and

$$\int S\sigma V\omega\rho dm = \int S\sigma\omega\rho dm = S . V\sigma\omega \int \rho dm$$

which vanishes for mass-centre. Therefore translation is conjugate to spin about mass-centre.

Momentum of $\alpha+\omega\beta$ is $m\beta+\omega\phi\alpha$, mass-centre being origin. If the screws coincide

$$\alpha+\omega\beta=x(m\beta+\omega\phi\alpha),\ \therefore\ \alpha=xm\beta,\ \beta=x\phi(xm\beta),$$

therefore β coincides with a principal axis, say $1=x^2m^2A$, then pitch $=\dfrac{1}{xm}=\pm\sqrt{A}$. This gives us six principal screws of the body.

APPENDIX II.

(A.)

SYLLABUS OF LECTURES ON MOTION.

Division of the Subject.

The science which teaches how to describe motion accurately, and how to compound different motions together, without considering the circumstances under which motions take place, is called *Kinematic* (κινημα, motion).

The simplest kind of motion is that in which a body without changing its size or shape moves so that all straight lines in the body remain parallel to their original positions; this motion is called a *Translation.* As all parts of the body move alike, we may confine our attention to any one of them, however small; for this reason that part of Kinematic which treats of translations is often called the Kinematic of Particles.

A body which does not change its size or shape during the time considered is called a *rigid* body. That part of Kinematic which treats of motions in which there is no change of size or shape is called the Kinematic of Rigid Bodies.

A change of size or shape, considered without reference to change of position, is called a *strain.* The Kinematic of Strains teaches how to describe strains accurately, and how to compound them together. Bodies which change their size or shape are called *elastic;* and the corresponding branch of Kinematic is called the Kinematic of Elastic Bodies.

The science which teaches under what circumstances particular motions take place is called by one or other of two different names according to the view that is taken of it. If it is regarded as mainly based upon the Law of Force, and if its results are expressed in terms of force, it is called *Dynamic* (δυναμις, force); but if it is regarded as mainly based upon the Law of Energy, and if its results are expressed in terms of energy, it is called *Energetic* (ἐνεργεια). In either case it is divided into two parts; *Static,* which treats of those circumstances under which *rest* or *null motion* is possible, and *Kinetic,* which treats of circumstances under which actual motion takes place. Properly speaking, Static is a particular case of Kinetic which it is convenient to consider separately.

We may also make divisions between the Static and Kinetic of particles, rigid bodies, and elastic bodies; but the Static of particles and of rigid bodies is generally treated as one subject, while the Kinematic and Dynamic or Energetic of elastic bodies are grouped together as the science of *Elasticity*.

These divisions may be represented by the following scheme:—

Science of Motion.

$$\left.\begin{array}{l}\text{Kinematic,}\\ \text{Dynamic, or}\\ \text{Energetic, viz.}\end{array}\right\}\left\{\begin{array}{l}\text{Static,}\\ \text{Kinetic,}\end{array}\right\}\text{ of }\left\{\begin{array}{l}\text{Particles (Translations),}\\ \text{Rigid Bodies (Twists),}\\ \text{Elastic Bodies (Strains).}\end{array}\right.$$

Translations.

Def. If two bodies A and B are in motion, the motion of B is said to be compounded of the motion of B relative to A, and the motion of A.

Prop. Translations represented by the sides of a parallelogram compound together into a translation represented by the diagonal.

Def. A *vector* is a quantity having magnitude and direction. A translation is a particular kind of vector, and the composition of translations is equivalent to their addition as vectors; it satisfies the law

$$\alpha+\beta=\beta+\alpha.$$

Def. Uniform rectilinear motion is that in which equal spaces are traversed in equal times.

Its equation is

$$\rho=\alpha+\beta t.$$

Prop. Two uniform rectilinear motions compound into a uniform rectilinear motion.

Harmonic Motion.

Def. Uniform motion in a circle is that in which equal arcs are traversed in equal times.

Def. If a point P move uniformly in a circle, and a perpendicular PM be always drawn from it to a fixed diameter AA' of the circle, the foot M of the perpendicular will oscillate to and fro in the diameter; this motion of the point M is called a *Simple Harmonic Motion*.

Its equation is

$$\rho=\alpha\cos(nt-\epsilon).$$

Def. The radius of the circle is called the *amplitude* of the s. h. m.

Def. The time which P takes to go once round the circle is called the *period* of the s. h. m.

Def. The circular measure of the arc described by P from the era of reckoning till it came to the positive end of the diameter AA' is called the *epoch* of the s. h. m.

Def. The portion of the whole period which has elapsed since the point M last passed through its middle position in the positive direction is called the *phase* of the S. H. M.

Prop. Two S. H. M. of the same period compound into a S. H. M. of that period.

The construction here made use of for compounding two S. H. M. is exemplified in the Tidal Clock of Sir W. Thomson. The clock has two hands whose lengths are proportional to the solar and lunar tides respectively, while their periods of revolution are made equal to the periods of these tides. A jointed parallelogram is constructed, having the hands of the clock for two sides; the height of that extremity of the parallelogram which is furthest from the centre will then be proportional to the height of the compound tide. For this purpose a series of horizontal strings at equal distances are stretched across the face of the clock, and the height is read off by running the eye along these to a vertical scale of feet in the middle.

Def. The curve described by a point which has a uniform rectilinear motion compounded with a S. H. M. perpendicular to it is called a *harmonic curve*.

The composition of S. H. M. of different periods in the same line may be represented graphically by the super-position of harmonic curves; i. e. by drawing a curve whose height at any point is the sum of their heights.

Prop. Any S. H. M. may be resolved into two in the same line, differing in phase by a quarter period, and one of them having any given epoch.

Prop. S. H. M. on any number of different lines, having the same period and phase, compound into one having that period and phase.

Prop. Two S. H. M. on different lines, having the same period, but differing in phase by $\frac{1}{4}$, compound into harmonic motion in an ellipse, (viz. an orthogonal projection of circular motion).

Its equation is

$$\rho = \alpha \cos(nt - \epsilon) + \beta \sin(nt - \epsilon).$$

Prop. Any number of S. H. M. having the same period compound into harmonic motion in an ellipse.

Two harmonic motions in different directions and with different periods produce a resultant which is best studied by wrapping round a cylinder of suitable size paper on which is traced a harmonic curve. The curve thus drawn on the cylinder may then be constructed in wire, and when turned round the axis of the cylinder will represent to an eye at a sufficient distance the curve of compound harmonic motion for varying values of the difference of phase of the simple motions. The simplest case is that in which the circumference of the cylinder is equal

to the length of a wave of the harmonic curve; here the periods are equal, and the curve traced on the cylinder is merely an ellipse. The same result is produced by turning the cylinder round its axis while a pencil moves with simple harmonic motion up and down a generating line.

DEF. A motion which exactly repeats itself in the same place after a certain interval of time is called a *periodic* motion.

The resultant of any number of simple harmonic motions whose periods are commensurable is a periodic motion, its period being the least common multiple of their periods.

Fourier's Theorem. Every rectilinear periodic motion of period P may be resolved into a series of simple harmonic motions whose periods are P, $\frac{1}{2}P$, $\frac{1}{3}P$, etc.

Let $\phi(t)$ be the distance of the moving point from a fixed point on the line at a time t, then the periodicity of the motion is expressed by the fact that $\phi(t+P)=\phi(t)$, whatever t is. And the theorem asserts that in this case the quantities a, b can always be found so as to make true the following equation, where

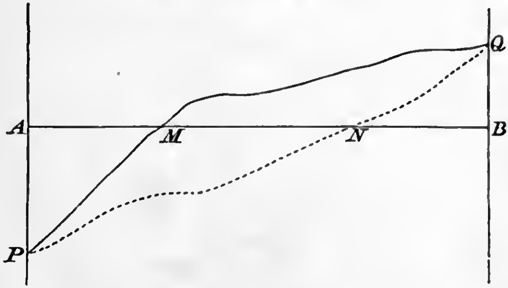
$$\theta=\frac{2\pi t}{P}\quad \phi(t)=\tfrac{1}{2}b_0+b_1\cos\theta+b_2\cos 2\theta+\dots$$
$$+a_1\sin\theta+a_2\sin 2\theta+\dots$$

The amplitudes and epochs of the several harmonic components may be represented as follows. Let a vertical cylinder revolve about its axis, while a pencil moves up and down one of its generating lines, so as to trace out a curve on the cylinder. If the motion of the pencil is periodic, and has a period equal to that of the cylinder or any exact multiple of it, this curve will return into itself and be a finite curve on the cylinder. Now let the pencil have the given periodic motion which it is required to resolve into harmonic constituents. When the cylinder revolves once in the period P, let the curve described be called C_1; when it revolves twice in that period let the curve be called C_2; when it revolves m times, let this curve be called C_m. And let a circle be drawn on the cylinder whose height is the mean height of the curve C_1; this will be called the mean circle.

If a plane be drawn through the axis of the cylinder, any curve traced on the cylinder may be orthogonally projected on that plane. It is necessary now to define the area between this projection and the line in which the plane is cut by the plane of the mean circle. Let AB be this line, and let $PMQNP$ be the projection, where PMQ is projected from the near half of the cylinder, and QNP from the further half. Then for the *near* half, the area APM which is *below* AB must be considered negative, and the area MQB which is *above* it, positive. For

the *further* half, QNB must be considered negative, and NPA positive. Thus the area is

$$-APM + MBQ - NBQ + APN$$
$$= MPN + MNQ = MPNQ.$$



The same rule is to be applied when the curve cuts itself or the line AB any number of times. Now it is found that for every closed curve traced on a cylinder, there is one plane through the axis such that the area of the projection on it is zero; and that for the plane at right angles to it the area is the greatest possible; while for an intermediate plane the area varies as the sine of the angle which it makes with the zero plane. It is thus possible to draw an ellipse upon the cylinder, the area of whose projection upon any plane whatever through the axis shall be the same as that of a given closed curve. Let the ellipse E_1 have the same projected area as the curve C_1, E_2 half that of the curve C_2, E_m one-mth that of the curve C_m, and so on. If, while the cylinder revolves once on its axis during the period P, the pencil be made to follow the ellipse E_1, always remaining in the same vertical line, it will have a S. H. M. with the period P. If while the cylinder revolves m times during the period P, the pencil be made to follow the ellipse E_m, it will have a S. H. M. with the period $\frac{1}{m}P$. These motions are the harmonic components of the given periodic motion; and that motion may be reproduced by compounding them all together*.

Parabolic Motion.

PROP. If rectilinear motion in which the space passed over from the beginning is proportional to the square of the time occupied, be compounded with the rectilinear motion, the resultant will be motion in a parabola.

Its equation is $\rho = \alpha + \beta t + \gamma t^2$.

[* Cf. *Dynamic*, p. 37, where it is said a proof of Fourier's Theorem will be given in the Appendix.]

Velocity.

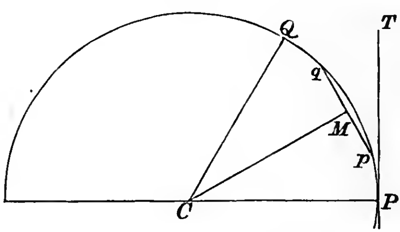
Def. If a body is in uniform rectilinear motion, and travels v centimetres in every second, the body is said to have at every instant a *velocity* of v centimetres per second, or simply a velocity v.

Def. If a body undergo a translation whereby a point of it is carried in any manner by any path from A to B in t seconds, the body is said to have a *mean velocity* $\frac{AB}{t}$ in that interval of t seconds.

The two quantities here defined have magnitude and direction; they are *vectors*. A velocity may be expressed in terms of other units than centimetres per second; in feet or miles per second, leagues per hour, etc.; but when expressed as a number of centimetres per second, it is said to be given in *absolute measure*. In uniform rectilinear motion the mean velocity is the same in any interval whatever, and is equal to the instantaneous velocity at any instant; but the latter is a property which the body possesses *at an epoch* or point of time, while the former is a fact relating to its motion during an interval.

Def. If any rectilinear motion of a point be compounded with a uniform motion of unit velocity at right angles to it, the curve traced out by the point is called the curve of positions for that rectilinear motion.

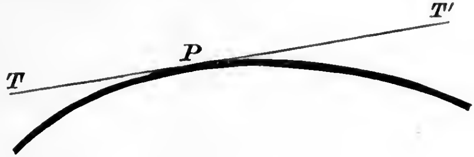
Lemma. PT is the tangent at a point P of a circle. Any angle being proposed, it is always possible to take a point Q on the circle so near to P that the chord of every arc pq included in PQ shall make with the tangent PT an angle less than the proposed angle.

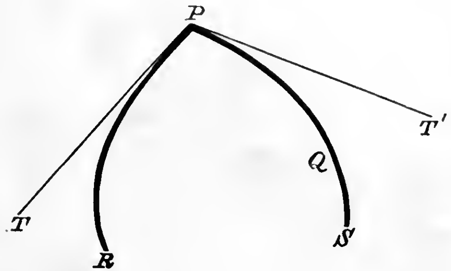


Let C be the centre of the circle; make PCQ less than the proposed angle, and draw CM perpendicular to pq. Then PCM is the angle which the chord pq makes with PT, and it is always less than PCQ, therefore less than the proposed angle. Q.E.D.

Def. R, P are points on any curve, Q moves from R along the curve towards P; if when any angle is proposed, it is always possible to take Q so near to P that the chord of every arc pq included in PQ shall make

with a certain line TP an angle less than the proposed angle; then the curve is said to have TP for a *tangent* at the point P.



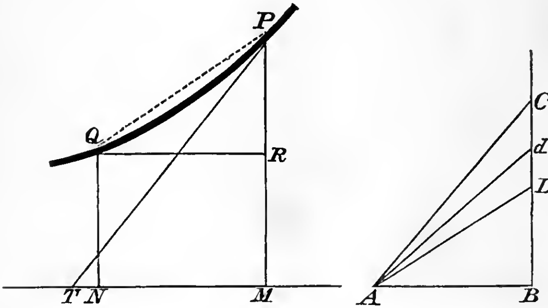


If S is a point on the other side of P and if Q moves from S towards P, there may be another line PT' such that an arc PQ may always be taken in which no chord shall be inclined to PT' so much as by a proposed angle. In this case we may speak of TP as the tangent *up to* P and of PT' as the tangent *on from* P. When TPT' is a straight line, the curve is said to be *elementally straight* or to have the property of *elemental straightness* at the point P; for the more it is magnified, the more will a portion containing P of given length in the magnified figure approach to the straight line TPT' in shape and position. For this, three conditions are necessary; there must be a tangent up to P, a tangent on from P, and these tangents must be in one straight line.

Prop. If the curve of positions of a rectilinear motion has a tangent at a point P, then it is possible to choose an interval ending at the instant corresponding to the point P so that the mean velocity in that interval (and in all intervals included in it) shall differ less than by a given amount from a certain quantity.

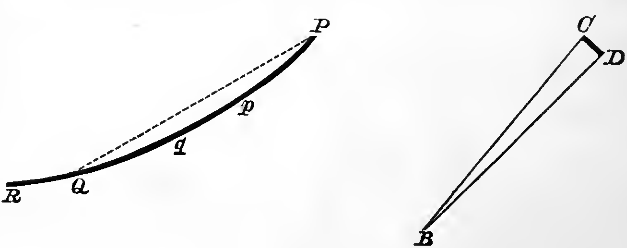
Let QP be a portion of the curve of positions, PT the tangent at P; QN, PM parallel to the (vertical) rectilinear motion considered, and perpendicular to the (horizontal) uniform motion with which it is compounded; QR perpendicular to PM. Since the uniform motion has unit velocity, the number of units of length in NM is equal to

the number of seconds in which the body has performed the vertical motion RP, and the mean velocity in the interval NM is therefore $\frac{RP}{NM}$.



Now take AB a horizontal line equal to the unit of length, and draw AC, AD parallel to PT, PQ, meeting the vertical line through B in C, D. Then BD represents the mean velocity in the interval NM. Similarly if pq be any arc included in PQ (pm, qn perpendicular to NM), and if we draw Ad parallel to the chord pq, Bd will represent the mean velocity in the interval nm. Now it is possible by hypothesis to choose Q so near to P that the angle QPT, which is equal to CAD, shall be less than any proposed angle; and that the angle which any chord pq makes with PT, which angle is equal to CAd, shall be less than the proposed angle. Therefore it is possible so to choose N that for every interval included in NM the length Cd shall be less than a proposed amount; or so that the mean velocity shall differ from the velocity represented by BC by a quantity less than the proposed amount. Q.E.D. The quantity Bc or MP/TM is then called the *instantaneous velocity* of the rectilinear motion at the instant M.

DEF. Let Q, P be successive positions of a moving point, and let BD represent the mean velocity during an interval included in the passage

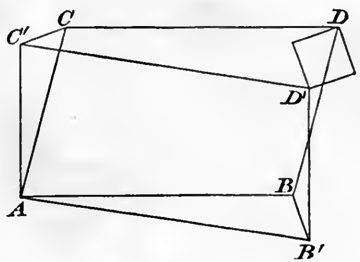


from Q to P; then if it is always possible to find Q so near to P that for all intervals between Q and P the distance DC from D to a fixed point C shall be less than a proposed length, the point at the instant of arriving at P is said to have an *instantaneous velocity* BC in magnitude and direction.

PROP. If a moving point has a velocity, the curve described has a tangent in the same direction; and if a length equal to the arc RQ be measured off on a straight line as Q moves, this rectilinear motion will have a velocity whose magnitude is equal to that of Q.

PROP. If each of two motions has a velocity at a certain instant of time, the motion compounded of them has a velocity which is compounded of their velocities by the rule for addition of vectors.

Let AB and AC be the given velocities; complete the parallelogram $ABDC$. Let also AB', AC' be the mean velocities during an interval



which ends at the given instant; if the parallelogram $AB'D'C'$ be completed, we know that AD' is the mean velocity of the resultant motion. Now the interval may be so chosen that for it and all shorter ones included in it BB' and CC' are each less than half of any proposed length; and therefore DD', which is their vector-sum, less than the proposed length. Consequently AD is the velocity of the resultant motion at the given instant. Q. E. D.

It is to be noticed that in accordance with our definitions a motion may have one velocity *up to* a certain instant and another velocity *on from* that instant; or, as we may say, an *arrival* and a *departure* velocity. Such motions are for mathematical convenience supposed to take place in the theory of collisions; but it is believed that they do not occur in nature, and that the arrival and departure velocities are always identical. If a point has an arrival and a departure velocity at a given instant and if they are identical, its motion is said to be *elementally uniform;* for if a small portion of the path containing the position of the point

at that instant be magnified to a definite length, and the times of traversing different parts of it be preserved in their proportions, then the smaller the portion taken, the nearer will the path approach to a straight line and the motion to uniform motion along it.

PROP. The velocity in the S. H. M.

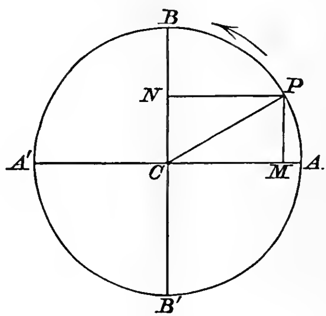
$$\rho = a \cos (nt - \epsilon)$$

is

$$\dot{\rho} = -na \sin (nt - \epsilon),$$

(when the position-vector of a point is called ρ, its velocity is denoted by $\dot{\rho}$).

The S. H. M. has a velocity, because its curve of positions has a tangent, being produced by unrolling an ellipse from a cylinder. Now uniform circular motion being compounded of two simple harmonic motions, its velocity is compounded of their velocities by the law of addition of vectors. Thus the velocity of P is compounded of the velocities of M and N; but these velocities are respectively perpendicular to the lines CP, CM, and MP, the vector CP being equal to $CM + MP$. The velocities are there-



fore proportional to the lengths of these lines, and as the velocity of P is $n \, . \, CP$ along the tangent, the velocities of M and N are $n \, . \, MP$ and $n \, . \, CM$ along MC and CN respectively. But a length $n \, . \, MP = n \, . \, AC \sin PCM$ along MC is equal to $-na \sin (nt - \epsilon)$. Q. E. D.

PROP. The velocity in the elliptic harmonic motion,

$$\rho = \alpha \cos (nt - \epsilon) + \beta \sin (nt - \epsilon),$$

is

$$\dot{\rho} = -n\alpha \sin (nt - \epsilon) + n\beta \cos (nt - \epsilon)$$

$$= n \left\{ \alpha \cos \left(nt - \epsilon + \frac{\pi}{2} \right) + \beta \sin \left(nt - \epsilon + \frac{\pi}{2} \right) \right\},$$

and is therefore proportional to the conjugate diameter.

PROP. The velocity in the parabolic motion

$$\rho=a+\beta t+\gamma t^2$$

is

$$\dot{\rho}=\beta+2\gamma t.$$

Let t_1, t_2, t_3, t be successive values of t, these quantities being therefore in ascending order of magnitude; $\rho_1, \rho_2, \rho_3, \rho$ the corresponding values of ρ. Then the mean velocity in the interval from t_2 to t_3 is

$$\frac{\rho_3-\rho_2}{t_3-t_2}=\beta+\gamma\,(t_2+t_3).$$

Since t_2 and t_3 are intermediate between t_1 and t, this vector differs from $\beta+2\gamma t$ less than $\beta+2\gamma t_1$ does; that is, less than $2\gamma\,(t-t_1)$. Now it is possible so to choose t_1 that this shall be shorter than any proposed length γx; that is, it is possible to choose an interval ending at t, so that the mean velocity for every interval included in it differs from $\beta+2\gamma t$ by less than a proposed amount. The same thing may be shewn for intervals beginning at t. Therefore the motion is elementally uniform and has $\beta+2\gamma t$ for its velocity. Q. E. D.

PROP. In the motion whose equation is

$$\rho=at^n$$

(n a positive integer), the velocity is

$$\dot{\rho}=nat^{n-1}.$$

With the notation of the previous proposition, the mean velocity in the interval from t_2 to t_3 is

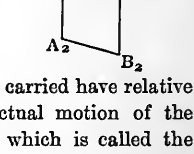
$$\frac{\rho_3-\rho_2}{t_3-t_2}=a\frac{t_3{}^n-t_2{}^n}{t_3-t_2}=a\,(t_3{}^{n-1}+t_3{}^{n-2}\,t_2+\ldots+t_2{}^{n-1}).$$

Since t_2 and t_3 are intermediate between t_1 and t, this quantity differs from nat^{n-1} less than $nat_1{}^{n-1}$ does; that is, less than $na\,(t^{n-1}-t_1{}^{n-1})$, which by proper choice of t_1 can be made less than an assigned quantity. Whence as before.

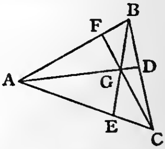
(B.)

I. Divisions of the Subject. Dynamic is the doctrine of the circumstances under which motion takes place. It is preceded by the description of motion, or Kinematic. The simplest motions are translations, with no change of aspect or shape; then motion of rigids, with no change of shape; then strains or changes of shape. In Dynamic we may consider circumstances of actual motion, or circumstances of possible rest; these are sometimes called Kinetic and Static; and these again may be of particles, rigids, or elastic bodies. In the latter the theory of fluids is specially distinguished.

II. Translations and their composition. The change of position or total effect produced by a translation is represented by a line of given length drawn in a given direction—*a vector*. It is determined by assigning three quantities. Quantities can only be described in words or numbers approximately; the right way of specifying a quantity (of length, time, or weight) is by drawing a line which represents it on a certain scale. A vector may then be specified by 3 lines marked off on the same scale; or by 2 lines in descriptive geometry.

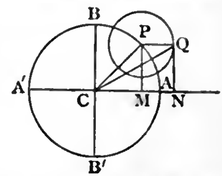


The composition of vectors arises out of relative motion. If any carriage or vehicle have the motion represented by one vector α, and a thing carried have relative to it the motion represented by another β, the actual motion of the thing carried is represented by a third vector γ, which is called the resultant or sum of the other two. This composition is represented by the sign $+$, thus $\alpha+\beta=\gamma$, or $AB+BC=AC$; this equation gives the rule of addition. If two motions be successively undergone by a body, the resultant change of position is their sum. In all cases $\alpha+\beta=\beta+\alpha$. Another rule: if $AD=DB$, $OA+OB=2OD$. If $AD:DB=q:p$, $(p+q)OD=p.OA+q.OB$. For triangle, $\overline{p+q+r}.OG=pOA+qOB+rOC$, if etc. Hence properties

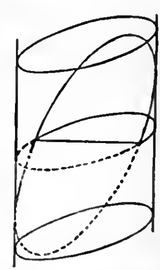


of centre of triangle and tetrahedron. Expression of vector in terms of 3 vectors: $ix + jy + kz$. Special representation on plane: complex numbers. Projection of lines on lines and planes.

III. Representation of Motion. Uniform Motion. The motion of a particle is specified when its path is laid down and when we are told at what point of this path it was at each instant of time. This may be done by a curve whose abscissæ represent the time and ordinates the distance traversed. In uniform motion equal distances are traversed in equal times: hence the distance is always proportional to the time. This is represented by a straight line. Also by the equation $s = vt$. Uniform rectilinear motion is represented by the vector equation $\rho = \alpha + \beta t$; uniform circular by equation $\rho = \alpha \cos nt + \beta \sin nt + \gamma$. Period, phase, amplitude, epoch. Two uniform rectilineal motions compound into uniform rectilinear; so two circular of same period and direction.

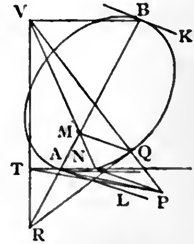


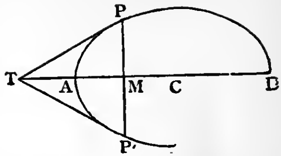
IV. Harmonic Motion. Definition, amplitude, period, epoch, phase, equation. Combination of 2 S.H.M. of one period in one line; tidal clock, etc. Curve of positions; construction by ellipse on cylinder. Properties of projection (orthogonal). Parallel lines remain parallel and are all altered in a certain ratio. Properties of ellipse so derived; centre, conjugate diameters, tangents at extremities of these axes. General projection of uniform circular motion is elliptic H. M., equation $\rho = \alpha \cos nt + \beta \sin nt$ where α, β are conjugate semi-diameters. Any S. H. M. of same period may be reduced to two sets, each set having same phase, but phases differing $\frac{1}{4}$; each set may be reduced to one S. H. M.; hence resultant is always E. H. M. Case of two S. H. M. variation of resultant with varying difference of phase. Revolving ellipse on cylinder.



Combination of different periods. Note and octave. Curves of position. Different directions; figure of 8. Method by curves on cylinder. Bird's wing. Periodic motion generally. Statement of Fourier's Theorem.

V. Parabolic Motion. Definition of parabola. Proof that abscissa $\propto$ (ordinate)^2. Path of motion $\rho = \alpha + \beta t + \gamma t^2$. Combination of Parabolic Motions with same or different axes.





VI. Direction of Motion (Tangents). Proof that in circle or ellipse $TA : TB = AM : MB$, and $CA^2 = CM . CT$. In parabola $AT = AN$. Harmonic situation of 4 points. Tangents at P, P' intersect on diameter bisecting PP'. In parabola $SP = ST = SG$. Solution of problem to draw parabola from P to Q having given tangent at P and direction of axis. All this is on assumption that tangency is unaltered by projection, or tangent regarded as line two of whose intersections coincide.

VII. Velocity. Of uniform rectilinear and circular motion. Velocity is directed quantity. Assumption that velocity may be compounded as vectors: true for two uniform rectilinear or circular motions. Velocity of S. H. M. Representation by curve of positions; tangent to the curve of sines. Problem the same as that of tangents. Velocity of parabolic motion, of elliptic H. M.

VIII. Hodograph. Acceleration. In Harmonic and Parabolic Motions.

IX. Fluxions. Exact definition of tangent and velocity. Proof of law of composition of velocity. Tangent of circle. Tangency retained in projection. Rate of variation of a quantity. Fluxion of at^n. Change of independent variable. Fluxion of $f(P, Q)$, etc.

X. The Inverse Method. Space traversed = area of curve of velocities. Wallis's integration of x^k.

XI. Curvature. Plücker's apparatus for plane. Stoppage of motion of point or line gives cusp or inflexion. Generally angular velocity ÷ linear velocity is curvature-mean and total curvature expression ρ''. Circle of curvature; construction for ellipse and parabola. Resolved parts of acceleration: fluxion of $\dot\rho (= v\rho')$ is $\ddot\rho = v^2\rho'' + \dot v\rho'$. Tortuous curve; point line and plane. Equation of helix. No acceleration perpendicular to osculating plane. Total curvature of tortuous curve. Tortuosity.

XII. Logarithmic Motion. Quantity equally multiplied in equal times increases at rate proportional to itself. Density of air. Intensity of light in water. $\dot s = ps . p$ called *intrinsic rate*. If such a quantity is ever zero, and varies continuously, it is always zero. If s is multiplied by P in every second, and $= a$ for $t = 0$ then it is $= aP^t$ whenever t is commensurable. P is function of p which changes to P^n when p changes to np. Let e be value of P when $p = 1$, then $s = ae^{pt}$ when p and t are commensurable; even when pt is commensurable. Extension of definition- ae^{pt} is to mean result of making a grow at intrinsic rate p for t seconds. Expansion in series. Prove that $e^x e^y = e^{x+y}$. Extension of $\dot s = ps$ to case

where s is vector in a plane, logarithmic spiral. Case of $\dot{s} \perp s$; circle; $e^{I\theta} = \cos\theta + I\sin\theta$, construction of e^{I} and e^{1+I}. Series of one sign, convergent if sum of n terms never exceeds fixed quantity M. In this case a limit exists, independent of order of terms. $\pm$ series, the four cases; if both divergent, the sum is arbitrary. Series for $\frac{1}{1-x}$, convergent for $x^2 < 1$; series $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ divergent. Exponential series always convergent. Fluxion of sum of series = sum of series formed by fluxions of its terms, except when convergence is infinitely slow.

XIII. Central Orbits.

1. Moment of velocity. Moment of resultant = sum of moments of components. Hence rate of change of moment of velocity = moment of acceleration. This is also $\dot{h}$ if h = twice rate of describing areas. Proof by second fluxion of $re^{I\theta}$ that $r \times$ transverse acceleration $= \dot{h}$. Newton's proof that if moment of acceleration = 0, then areas are proportional to time. In central orbit $r^2\dot{\theta} = h$, therefore angular velocity is inversely as square of distance.

2. Related curves. Inverse: cut radius vector at equal angles. Description by Peaucellier's cell and Hart's cell. Pedal; construction of tangent by circle. Reciprocal. Reciprocal of circle is conic.

3. Hence motion in conic with acceleration tending to focus has circular hodograph. Acceleration proportional to angular velocity, $\therefore$ to $\frac{1}{r^n}$. Compare with Kepler's laws. Two constant parts of velocity.

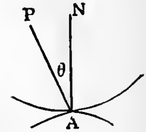
4. Equation of Energy $\partial_s(\frac{1}{2}v^2) = \partial_t(v) = P\partial_s r$, $\frac{1}{2}v^2 = \frac{\mu}{r} - \frac{\mu}{2a}$ in elliptic motion. Velocity from infinity at any point. Case of parabola.

5. Any orbit, $\rho = p + \partial^2_{\phi}p$ in hodograph gives $\frac{P}{hu^2} = (1 + \partial^2_{\theta})\, hu$ in orbit. So $\rho = \frac{r\dot{r}}{\dot{p}}$, $\frac{v^2}{\rho} = P\frac{p}{r}$, $vp = h$ gives $P = \frac{h^2\dot{p}}{p^3\dot{r}r}$. Apply to $r^m = a^m\cos m\theta$, here $a^m p = r^{m+1}$, $P = (m+1)\,h^2a^{2m}r^{-3-2m}$. These curves got from $\xi + \iota\eta = (x + \iota y)^m$: Examples.

XIV. Motion of Plane on Plane. General Theorems. Any finite change of position may be produced by rotation. Case of translation belongs to infinitely distant centre. Direct and inverse triangles. Composition of two finite rotations by inverse triangles. Hamilton's theorems about inverse polygons. Effect of any number of successive rotations represented by rolling of polygons. Limiting case; any motion of plane can be represented by rolling of curves.

Theorem of the instantaneous centre. System of velocities; their distribution. $\dot{\rho} = Ia\rho$ with rotational velocity a. Composition of such systems:

rotational velocity $a, b, c\ldots$ at points $A, B, C\ldots$ have resultant $a+b+c\ldots$ at point G such that $(a+b+c+\ldots)\ OG = a\,.\,OA + b\,.\,OB + c\,.\,OC + \ldots$ Two equal and opposite rotations give translation. Composition of translation with rotation.



In rolling, angular velocity = sum of curvatures × velocity of point of contact. Velocity and acceleration of every point in plane, locus of points of zero normal or tangential acceleration is a circle. Curvature of roulette. [Velocity of $P = I\,.\,AP\,(\dot\phi + \dot\psi)$. Hence acceleration $= I\,.\,A\dot{P}\,(\dot\phi + \dot\psi) + I\,.\,AP\,(\ddot\phi + \ddot\psi)$ but $\dot{AP} = \dot{P} - \dot{A} = I\,.\,AP\,(\dot\phi + \dot\psi) - \dot{A}$, ∴ acceleration $\ddot{P} = -AP\,(\dot\phi + \dot\psi)^2 - I\dot{A}\,(\dot\phi + \dot\psi) + I\,.\,AP\,(\ddot\phi + \psi)$. Curvature of path of

$$P = \frac{\text{normal acceleration}}{\dot{p}^2} = \frac{-AP\,(\dot\phi + \dot\psi)^2 + \dot{A}\cos\theta\,(\dot\phi + \dot\psi)}{AP^2\,(\dot\phi + \dot\psi)^2}$$

$$= -\frac{1}{AP} + \frac{\cos\theta}{AP^2}\cdot\frac{rr'}{r+r'}.]$$

XV. Special Cases. Combination of circular translation with rotation = rolling of circle on circle. Case of internal rolling, radii as 1 : 2. Double generation of circular roulettes: envelop of diameter. Tricusp, cardioid, quadricusp. General theorem for any three curves.

Circle on straight line. Curvature of cycloid. Theory of involute and evolute. Arc of cycloid and epicycloid. Three-bar motion. Inverted parallelogram; rolling of equal conics. Locus of point rigidly attached to bar is inverse of conic. Kite.

XVI. Motion of sphere on sphere. Analogues of Theorems in XIV and XV. Rotational velocities about axes which meet combined by addition of vectors. Theorem of moments.

XVII. General motion of Rigid Body. Any change of position may be produced by a twist. Erroneous proof that it may be produced by rotation. Two congruent figures $ABC\ldots K$, $A'B'C'\ldots K$, if $Oa = AA'$, $Ob = BB'\ldots Ok = CC'$ will be directly congruent if $abc\ldots k$ has even dimensions, inversely if it has odd.

Instantaneous motion is twist-velocity. If two systems of velocities are consistent with rigidity, so is resultant system. Any number of twists compound into single twist. Composition of two twists: cylindroid. Resolved part of motion of any point of line along line $= k\sin\theta - p\cos\theta$. Complex of screw, plane belonging to each point, and vice versa. Resolution of twist into two rotations, axis of one given. Two rotations to meet two given lines of the complex: viz. the tractors of the two lines and their co-axes. Sum of motions of pair of axes along their lengths due to unit rotation about any line $= k\sin\theta - p\cos\theta$, whichever the two axes are. Hence sum of such motions due to twist about second screw $= k\sin\theta$

$-(p+q)\cos\theta$. If this vanishes, a pair of axes of one screw are lines of complex of the other, and the screws may be said to *meet*. Otherwise the quantity is the *moment* of the screws.

XVIII. Strains. 1. String or rod: elongation or shortening; ratio $= a$, amount $= a - 1$. Homogeneous stretch.

2. Membrane; homogeneous strain may always be produced by two longitudinal strains at right angles. Ellipse of strain. Function of vector, $ix + jy$ becomes $ax + \beta y$. Shear; amount $= a - \frac{1}{a}$ or nearly $= 2\lambda$ if $a = 1 + \lambda$.

3. Solid; homogeneous strain produced by three longitudinal strains at right angles. Strain-ellipsoid. Vector function $\phi(\rho)$. Resolution of strain into elongation, shear, and uniform dilatation.

(C.)

BOOK III. STRAINS.

CHAPTER I. STRAIN-STEPS.

Strains in straight line. Strains in plane. Displacement conic. Linear Function of a Vector. Shear. Composition of Strains. Resultant. Product. General strain of Solid. Displacement-quadric. Composition of uniform strains. Non-uniform strain in terms of line-flux of displacement.

CHAPTER II. STRAIN-FLUX.

Strain-flux due to given velocity-system. Instantaneous spin. Irrotational velocity-system. Lines of flow and orthogonal surfaces. Velocity-system consistent with constant volume.

BOOK IV. FORCES.

CHAPTER I. THE LAWS OF MOTION.

Mass. Acceleration due to strain of adjacent body. Gravitation. Attraction and repulsion of electrified bodies, and of magnets. Law of composition. Resultant mass-acceleration of particle. Attwood's Machine. Law of Action and Reaction. D'Alembert's Principle. Effect of forces on rigid body. Parallelogram of forces. Analogy to Spins. Wrench. Work and Energy. Work of Twist against Wrench.

CHAPTER II. THE CONDITIONS OF EQUILIBRIUM OF A RIGID BODY.

Two forces. Three forces. Application to problems. Four or five forces. Tractors. Six forces. Lines in involution. Screw determined by five lines. General conditions. In passing through equilibrium no work is being done.

CHAPTER III. THE COMPOSITION OF FORCES.

SECT. 1.

The link-polygon. Couples. Relation of link-polygons with different poles. Reciprocal Diagrams. Parallel forces. Construction of Moment. Centre of parallel forces.

SECT. 2.

Centre of inertia: triangle, quadrilateral, tetrahedron, pyramidal frustum, circular arc, sector, segment, parabolic frustum, paraboloid, hemispherical surface. Second Moment: neutral axis and plane, swing-conic and swing-quadric. Core of parallelogram, triangle, ellipse, tetrahedron, ellipsoid.

SECT. 3.

Attraction and Repulsion inversely as square of distance. Spherical Shell. Potential and level surfaces. Analogy with motion of liquid. Sources and sinks. Lines of force. Theorems of Stokes and Chasles. Electric images. Centrobaric bodies.

CHAPTER IV. MOTION OF A RIGID BODY.

Momentum and energy of rigid body. Momental ellipsoid. Pendulum. Solution by elliptic functions. Motion under no forces. Poinsot's representation. Euler's equations. Solution by elliptic functions. Sylvester's theorem. Motion of Top. Spherical pendulum. Motion of Hoop. Precession and Nutation.

BOOK V. STRESSES.

CHAPTER I. SOLIDS.

Elastic string in straight line. Flexible inextensible string. Catenaries. Wire. Kirchhoff's theorem. Elastic curve. Spiral Spring. Horizontal rod slightly bent. Continuous girder. Plate strained in its own plane. Stress-conic. Plate bent. Solid strained in any way. Stress-quadric. Relation between stress and strain for isotropic solid. Crystals. Energy of strain. Variation of stress under forces. Expression in terms of displacement.

CHAPTER II. FLUIDS.

Equilibrium of Fluids. Level surfaces. Floating bodies. Metacentre. Molecular theory of Fluids. Theorems of Maxwell and Boltzmann. Elasticity and specific heat. Surface tension. Elastic curve. Motion of solid in a frictionless liquid. Vortex-lines and filaments.

CHAPTER III. WAVES AND VIBRATIONS.

Wave-transmission along flexible string. Straight tube of air. Shear-wave on rod. Tones of strings and pipes. Fundamental vibrations of a system.

(D.)

ELEMENTS OF DYNAMIC.

BOOK I.

Kinematic of Translations.

BOOK II.

Kinematic of Rigid Bodies.

BOOK III.

Strains.

BOOK IV.

FORCES.

BOOK V.

STRESSES.

BOOK VI.

HEAT.

BOOK VII.

WAVES AND VIBRATIONS.

APPENDIX III.

EXERCISES.

BOOK I. CHAPTER I.

1. Define a *rigid body*, and a movement of *translation*. Explain how translations are compounded together.

Find the locus of a point P which moves so that the length of the resultant of the translations PA, PB, PC is constant—the points A, B, C being fixed.

2. A leech crawls by alternately lengthening and shortening itself, holding fast by its head when it shortens, and by its tail when it lengthens. Describe this motion in kinematical language, analyzing it into its constituent parts.

3. What is meant by compounding translations together? Show from your definition that a change in the order of composition makes no difference in the result.

A, B, C, D, E, F are the vertices of a regular hexagon, and O is any seventh point. Find the resultant of the translations AO, OB, OC, OD, OE, OF.

4. Explain the equation of uniform rectilinear motion $\rho = \alpha + \beta t$.

Two points are moving uniformly in straight lines AB and CD, and in the same second they get from A to B, and from C to D respectively. Find by construction the positions in which they are nearest together.

5. Define a simple harmonic motion, its period, amplitude, and epoch. Prove that the resultant of any number of simple harmonic motions of the same period is motion in an ellipse.

The motion of a point is compounded of two simple harmonic motions at right angles to one another which are very nearly equal in period, but whose amplitudes are slowly diminishing at a uniform rate; find the general shape of the curve which the point will describe.

6. Define a "simple harmonic motion," its "amplitude" and its "period," and show that the resultant of two simple harmonic motions of the same period is in general an elliptic harmonic motion. What special cases are included in this description?

7. Draw some of the figures produced by compounding two simple harmonic motions in directions at right angles to one another, the periods being (α) as 2 to 1, (β) as 3 to 2.

8. State the experimental laws of motion, and define *force*. What two kinds of force are there? and which do you think more likely to be explained as a case of the other?

What is the difference between pressure due to contact with a strained body, and "action at a distance"? Mention any hypotheses by which it has been attempted to describe either of these as a case of the other.

9. Define a motion of *translation* of a rigid body, and explain what is meant by the composition of translations.

Three translations are represented by lines parallel and equal to the sides of a triangle; discuss the different possible values of their resultant.

10. Any number of simple harmonic motions in one plane and of one period compound into harmonic motion in an ellipse.

To two points A, B in the ceiling are fastened the ends of a string slightly longer than AB; and from the middle of the string a ball is hung by another string reaching nearly to the floor; if the ball be set in motion, what will be the nature of its path?

11. Define a simple harmonic motion; and show that the resultant of two such motions, having the same period, in the same straight line, is a simple harmonic motion of that period.

If the two motions differ by a quarter phase, prove that the squared amplitude of the resultant is equal to the sum of the squared amplitudes of the components.

Two simple harmonic motions take place 140 times and 150 times a second respectively; find how many times a second the amplitude of the resultant goes through all its changes.

12. A pencil moving with S.H.M. on a generating line of a cylinder which revolves uniformly in the same period will trace out an ellipse upon the cylinder.

Explain how curves so drawn upon a cylinder may be used to represent the composition of harmonic motions at right angles to one another; and describe the curves produced when the cylinder makes (1) half a revolution, (2) two revolutions in the period of the S.H.M.

13. Define a simple harmonic motion; and draw a curve of velocities for the compound of such a motion and its octave, the latter starting at phase $\frac{3}{4}$ when the former starts at phase 0.

CHAPTER II.

1. A point moves uniformly round a circle while the centre of the circle moves uniformly with less velocity along a straight line in its plane; find the nodes of the curve which the point describes.

2. Assuming the law of composition of velocities, and some rule for drawing a tangent to a parabola, find the velocity at any instant of a point moving with the horizontal component of its motion uniform in a parabola whose axis is vertical.

3. Find the velocity at any time in the parabolic motion

$$\rho = a + \beta t + \gamma t^2,$$

and hence show that there is a point where the tangent to the curve is perpendicular to γ.

The equation $\rho = (a + it)^2 a$, where i is the operation of turning counter-clockwise through a right angle, represents a parabolic motion with the origin for focus.

Hence, or in any other way, prove that the velocity in a parabolic motion is that due to a fall from the directrix.

4. If $\rho = a + \beta t^n$, prove that $\dot{\rho} = n\beta t^{n-1}$ when n is a positive integer.

5. Explain what is meant by the "hodograph" of a given motion, and find the hodograph in the case

$$\rho = at + \beta t^2 + \gamma t^3.$$

6. Find the normal and tangential accelerations of a moving point.

Find the curvature at any point of a parabola or of a cardioid.

7. Explain the equation $\epsilon^{i\theta} = \cos\theta + i\sin\theta$, and represent the series for ϵ^a, where $a = 1 + \frac{1}{2}i$, by a geometrical construction.

From the equation $\rho = r\epsilon^{i\theta}$, determine the radial and transversal accelerations of a point moving in a plane curve.

8. Prove that the acceleration of a moving point consists of a tangential part, which is the rate of change of the magnitude of the velocity, and of a normal part, which is the square of the velocity multiplied by the curvature of the path.

A train goes at 30 miles an hour round a curve of half-a-mile radius; find the deflection of a plumb-line hung in one of the carriages.

9. If the acceleration of a moving point is always directed towards a fixed point and inversely as the square of the distance from it, prove that the hodograph is a circle.

What is the form of the orbit when this circle is at a distance from the centre of acceleration very large compared with its radius?

Conversely, if the hodograph is a circle, and the acceleration is directed to a fixed centre, prove that it must vary as the inverse square of the distance.

10. Prove that uniform velocities are compounded according to the same rule as translations.

11. What are the *dimensions* of a velocity?

A molecule of air moves at the rate of twenty miles a minute; and light travels at the rate of 333 million kilometres per second; express each of these velocities in terms of the units here used to express the other, assuming that

$$8 \text{ kilometres} = 5 \text{ miles}.$$

12. What sort of motion has a velocity at a given instant? and how is that velocity defined? Describe the "curve of velocities," and show how it represents (1) the acceleration, (2) the space passed over.

13. Prove that the motion represented by the equation

$$\rho = \alpha + \beta t + \tfrac{1}{2}\gamma t^2$$

is one of uniform acceleration, and that the path described is a parabola. Prove in any other way that the path described under uniform acceleration is a parabola.

14. If a point move in a parabola whose axis is vertical with uniform horizontal velocity, show that the vertical component of its velocity increases uniformly with the time.

Hence, or in any other way, show that if in rectilinear motion the distance passed over is proportional to the square of the time, the velocity will be proportional to the time simply.

15. In the elliptic harmonic motion

$$\rho = \alpha \cos nt + \beta \sin nt;$$

show that the acceleration $\ddot{\rho} = -n^2\rho$.

A small bullet is fastened to the end A of a stiff elastic rod AB without mass, and it is observed that when the end B of the rod is held horizontally, the bullet weighs down the end A an inch and a half. The whole is then placed on a smooth table, and the end B held tight. Prove that the bullet will oscillate horizontally $8/\pi$ times in a second, the acceleration of gravity being 32 feet a second per second, and the acceleration of the bullet due to the elasticity of the rod being supposed proportional to the distance from its mean position.

16. A point moves in a parabola so that its distance from the axis increases uniformly from zero; show that its distance from the tangent at the vertex varies as the square of the time.

The points A and B move in two different parabolas according to this law, and the line AB is continually trisected in C and D; prove that the points C and D move in two other parabolas, and find the relation between the axes of the four curves.

17. State and prove the rule for composition of velocities.

A, B, C are points on an ellipse whose centre is O, and P is any point in its plane. Wherever P is, velocities represented by AP, BP, CP have always a resultant in the direction of OP. Prove that BC is parallel to the tangent at A, and that the area of the triangle ABC bears a constant ratio to the area of the ellipse.

18. Define Acceleration, Mass, Force. State what is the approximate relation between mass-acceleration and position in the following cases:—

(1) A body falling freely in the neighbourhood of the earth's surface.

(2) A ball on a smooth table attached to the floor by an elastic string which passes through a hole in the table and is just unstretched when the ball is directly over the hole.

(3) The Moon.

19. If a particle moves in an ellipse under a force directed to one focus, prove that the hodograph is a circle and the acceleration inversely as the square of the distance.

20. Prove, by twice differentiating the vector of a moving point, that its total acceleration is compounded of accelerations $\frac{v^2}{r}$ and $\frac{dv}{dt}$ along the principal normal and the tangent to its path respectively.

21. Assuming that, for values of x between $-\pi$ and π, any function $f(x)$ of x which is such that the curve $y=f(x)$ has an area between those limits may be expanded in a series of the form

$$f(x)=\tfrac{1}{2}b_0+b_1\cos x+b_2\cos 2x+\dots\dots\dots$$
$$+a_1\sin x+a_2\sin 2x+\dots\dots\dots,$$

prove that π times any one of the coefficients a, b is a projection of that area on a diametral plane of a cylinder round which it can be wrapped a certain number of times.

If $f(x) = \sin x$ from 0 to $\frac{\pi}{2}$, and $=\cos x$ from $\frac{\pi}{2}$ to π, expand it in a series of sines which shall be true between 0 and π.

22. Define the velocity and the acceleration of a particle moving in a straight line. If a curve be constructed whose ordinates represent the velocities at times represented by the abscissae, in what way does the figure indicate the acceleration and the space described?

If
$$s = at^2 + b\cos(nt - a),$$
find the velocity and acceleration at any time.

23. How are velocities compounded? Define the total acceleration of a moving particle and the hodograph of its path. If a particle move uniformly in a circle, show that the hodograph is also a circle uniformly described; and find the magnitude and direction of the acceleration at any time.

24. State the second law of motion; and deduce from it that the path of a projectile is nearly a parabola. If two particles are projected in the same horizontal direction from two points not in a vertical line, find the relation between their velocities of projection that the paths may touch, and the times of arriving at the point of contact.

25. In what ways does the mass-acceleration of a body depend on its position relatively to other bodies?

If the mass-acceleration of a particle is directly proportional to its distance from a fixed point and directed towards the fixed point, show that its motion will be compounded of two simple harmonic motions in lines at right angles to one another.

BOOK II. CHAPTER I.

1. A rigid body has a given twist-velocity about a given screw. Find the velocity of any point in the body.

What is meant by the composition of two twist-velocities?

Rotations about the axes A, B, C can be compounded into a rotation about the axis D. Show that, when looked at from any point of D, the lines A, B, C will appear to meet in a point.

2. Prove that any change in the position of a plane figure in its plane may be produced by rotation about some point. What happens when the point is at an infinite distance?

Four rods AB, BC, CD, DA are jointed together, the length AB being equal to CD, and BC to DA. The two longer rods cross one another. If AB be fixed, prove that the motion of CD may be produced by the symmetrical rolling of a conic upon an equal fixed conic.

3. A plane slides on a fixed plane so that a fixed point of each lies on a fixed line of the other, find the locus of the instantaneous centre on the fixed and moving planes, and the locus of points which at any instant have no tangential acceleration. The motion of one plane on another is determined when two given curves on the moving plane touch two given curves on the fixed plane.

Investigate analogous determinations of the motion in space of a solid body which has one, two, or three degrees of freedom.

4. What relations must hold of the relative positions of n axes in order that small rotations about them may be equivalent to rest? ($n=3$, 4, 5, 6.) In the case $n=5$, show that, if one axis touch the hyperboloid containing three others, each axis touches the hyperboloid containing any three others.

CHAPTER II.

1. When a body has a given spin of magnitude ω, find the locus of those points in it which have a velocity of given magnitude v.

2. When a body has a twist about a certain screw, find the locus of those points whose velocity is in a given direction.

3. Two spins of equal magnitude about non-concurrent axes are compounded into a twist. Prove that its axis is equidistant from and equally inclined to the axes of spin.

4. Spins about the sides of a triangle and represented by their lengths taken in order, compound into a translation represented by twice the area of the triangle.

5. Extend this proposition to a plane or skew polygon.

6. If a line A revolve about a line B with angular velocity ω, prove that the lengthwise velocity of A is $k\omega \sin\theta$, where k is the shortest distance and θ the angle between A and B.

CHAPTER III.

A circle rolls uniformly inside another circle of double the radius; show that every point rigidly connected with it moves in an elliptic harmonic motion, and that the sum or difference of the axes of all the ellipses is constant.

BOOK III. CHAPTER I.

1. In a flat board subjected to uniform stress in its plane, if an ellipse can be found such that the pressure across each axis is represented in magnitude and direction by the other, prove that the same property belongs to any two conjugate diameters.

2. A regular hexagon undergoes a pure shear by sliding parallel to one of its sides. Find the amount of the shear that the deformed hexagon may have two right angles, and determine its other angles.

3. Any system of forces is equivalent to a wrench about a certain screw.

Prove that a rotating body will do no work against this wrench if

$$a \cos \phi = k \sin \phi,$$

where a is the pitch of the screw, k the shortest distance between the axis of the screw and the axis of rotation, and ϕ the angle between these axes.

4. In the case of a plane homogenous stress, show that a conic may be drawn such that the stress across any diameter is represented in magnitude and direction by the conjugate diameter.

5. Prove that a solid body can be moved from any one position to any other position by a twist about a certain screw. What becomes of the screw in the cases of pure translation and pure rotation?

A body revolves uniformly about an axis so as to make one revolution during the time in which it moves uniformly 2 feet along the axis. Find the locus of those points the direction of whose motion makes an angle of 60^0 with the axis.

6. Define a simple shear, the amount of a shear, the ratio of a shear. If the shear be small, show that the amount is twice the elongation or contraction.

7. Prove that the deformation at any point of a strained body is compounded of a dilatation or compression and two simple shears.

BOOK IV. CHAPTER I.

1. On the latus rectum of a parabola, as diagonal, a square is described; find the centre of gravity of that portion of the square which is inside the parabola.

2. For every plane area there exists an ellipse, such that if the area be subject to pressure at every point of it proportional to the distance of the point from a certain line in its plane, the resultant pressure acts at a point R, such that if G is the centre of gravity of the area, and P the pole of the line in regard to the ellipse, $RG = GP$, and RGP is a straight line.

CHAPTER II.

Find swing-radii in the following cases:

1. A right circular cylinder about its axis.

2. A right circular cylinder about a line cutting axis at right angles.

3. A right circular cylinder about a tangent to the circular rim.

4. A sphere about any tangent.

5. A regular tetrahedron about any edge.

6. A regular tetrahedron about a line bisecting two opposite edges.

7. A paraboloid of revolution in regard to a plane touching it at the vertex.

8. A paraboloid of revolution in regard to its base.

9. Determine the swing-conic of three particles of equal mass; and show that so far as second moments are concerned any area may be replaced by three such particles.

---

1. Define a *homogeneous strain*, a *pure strain*, a *simple shear*. Show how to represent the pure strain of a plane area by means of a certain ellipse, such that the new position of any diameter of the ellipse is perpendicular to the tangents at its extremities.

What kind of strain would take place if the same construction were made with a hyperbola?

2. Define the centre of parallel forces, and find it (1) when the forces are distributed uniformly over the surface of a quadrilateral: (2) when they are distributed over the surface of a triangle, proportionally to the distance from one side of it.

3. A regular hexagon is made of equal and uniform rods jointed together, and the ends of two opposite sides are joined by two parallel strings; the whole framework is then allowed to hang from one of these sides, which is held horizontal. Suppose (1) that the strings are inextensible, and find their tension; (2) that either of them would be stretched to twice its natural length by the weight of the six rods, and find the position of equilibrium.

4. Find the momentum and the kinetic energy of a rigid body rotating about a fixed axis.

When a pendulum performs small oscillations under the influence of gravity, prove that the centres of oscillation and suspension are convertible.

5. In a fluid at rest under the action of any forces, prove that the level surfaces are also surfaces of equal pressure and density.

If two tubes were constructed so as to lie along a meridian, one always

half-a-mile above the level of the sea, and the other always half-a-mile below, in what direction would water flow in these two tubes respectively?

6. A man stands with his feet on the upper bars of the backs of two chairs whose seats are turned away from him. His legs are straddled at an angle of 60^0. Find the conditions of equilibrium:—(1) regarding the man as a rigid body, (2) assuming that his legs are jointed freely at the hip for lateral motion.

7. Explain the operation of "getting up swing." Is this possible if you are placed perfectly at rest in a seat freely suspended by two equal ropes from points in the same horizontal plane? A trapezist who is swinging with a range of 30^0 on either side of the vertical, can increase the distance of his centre of inertia from the axis of oscillation by one eighth; show how by graphical construction to determine the number of swings in which he can increase his range to 60^0.

8. In a plane lamina under homogeneous stress in its plane, prove that a conic may always be found such that the stress across any diameter is represented in magnitude and direction by the conjugate diameter.

9. If a rigid body be in motion, show that at any instant the straight lines which, if rigidly connected with it, would have no longitudinal motion, form a linear complex.

Investigate the geometrical relations between a system of forces and the small motion by which no work is done on the forces.

10. State the Law of Energy for a single particle under the action of gravity. Describe Attwood's Machine; and show how the definition of mass derived from it enables us to extend the law of energy to two or more bodies. What motion can be communicated to a ton of material by the fall of 5 lb. through 160,000 ft.?

11. Define Rest and Equilibrium. Can a body have either without the other? "When a system of bodies is passing through a position of equilibrium, there is no change of kinetic energy, and therefore no work is being done." Explain this statement, and show that it amounts to the principle of virtual velocities. Four uniform rods, AB, BC, CD, DA, are jointed together at A, B, C, D, and placed in a vertical plane with AB resting on the ground. $AB=BC=DA=\frac{1}{2}\,CD$. A load λ times the weight of CD is placed upon it; find what positions of the load exert the greatest and least compression upon AB.

12. Give a construction for finding the resultant of a number of forces given in magnitude and direction by lines drawn on a piece of paper. When the forces are all parallel, in what way does this construction determine their moments about any point in the plane?

NOTES.

p. 51, l. 20, "its velocity," *i.e.* the velocity of the mass-centre.

p. 57, ll. 2 and 4 from bottom, "*immediately* before" and "*immediately* after"

p. 60, l. 1. This statement seems questionable—mass-acceleration $= m\frac{dv}{dt}$, rate of change of momentum $= \frac{d}{dt}(mv)$, and the two are not always equal.

p. 61, l. 6 up. I have received the following remarks on this paragraph: "Cut a perfectly homogeneous spherical shell of indian-rubber, so that there is no strain, and turn it inside out. This can be done if a small piece be removed. Surely the body would have *surface-strain.* Does not Clifford mean normal surface stress?"

p. 61, l. 11 up, "compression of disc" strikes one as being obscurely worded though the meaning is clear.

p. 62, l. 8, "stress across section," *i.e.* per unit area.

p. 66, no definition given of strength.

,, l. 21, for "*battery*" should we not read "cell"?

p. 71, introduce O at lower angle of figure.

p. 74, l. 16, rather "look back along it," *i.e.* in opposite sense to spin-rotor.

p. 76, l. 6 from bottom. Supply "Vorlesungen über Math. Physik."

p. 78, l. 10 up, "$T+T'$," the reference is, of course, to a well-known text-book.

INDEX.

VOL. I.

INDEX.

VOL. II.

CAMBRIDGE : PRINTED BY C. J. CLAY, M.A. & SONS, AT THE UNIVERSITY PRESS.